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On a conjecture about relative lengths

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We need some definitions from [1]. Let $\mathcal{C} \subset \mathbb{R}^2$ be a convex body. A chord pq of $\mathcal{C}$ is called an affine diameter of $\mathcal{C}$, if there is no longer parallel chord in $\mathcal{C}$. The ratio of $|ab|$ to $\frac{1}{2}|a'b'|$, where $a'b'$ is an affine diameter of $\mathcal{C}$ parallel to ab, is called the *$\mathcal{C}$-length* of ab, or the *relative length* of ab, if there is no doubt about $\mathcal{C}$. We denote it by $\lambda_{\mathcal{C}}(ab)$.

Denote by λ_n the relative length of a side of the regular n-gon. For every $ab \subset \mathcal{C}$ we have $|ab| \le |a'b'|$, where $a'b'$ is the affine diameter parallel to ab, hence $0 < \lambda_n = \frac{|ab|}{|a'b'|/2} \le 2$. For every regular triangle (or square), since its side length equals its corresponding affine diameter, $\lambda_3 = \lambda_4 = 2$. Let $\mathcal{C} = abcde$ be a regular pentagon with side length 1, join the points c and e, then we know that ab is parallel to ce and $\lambda_5 = \lambda_{\mathcal{C}}(ab) = \frac{|ab|}{|ce|/2} = 1/\cos(\frac{\pi}{5}) = \sqrt{5} - 1$ (see Fig. 1). Let $\mathcal{C} = abcdef$ be a regular hexagon with side length 1, join the points c and f, then ab is parallel to cf and $\lambda_6 = \frac{|ab|}{|cf|/2} = 1$ (see Fig. 2).

A side ab of a convex n-gon $\mathcal{P}$ is called *relatively short* if $\lambda_{\mathcal{P}}(ab) \le \lambda_n$, and it is called *relatively long* if $\lambda_{\mathcal{P}}(ab) \ge \lambda_n$.

Sind $\mathcal{C}$ eine konvexe Figur und ab eine Strecke der Euklidischen Ebene, so wird im nachfolgenden Beitrag das Verhältnis der Länge $|ab|$ zur Hälfte der Länge einer längsten Sehne von $\mathcal{C}$ untersucht, die parallel zu ab ist. Dieses Verhältnis wird relative Länge von ab genannt und mit $\lambda_{\mathcal{C}}(ab)$ bezeichnet; die relative Länge einer Seite eines regelmässigen n-Ecks wird durch λ_n abgekürzt. Beispielsweise gilt $\lambda_3 = \lambda_4 = 2$, $\lambda_5 = \sqrt{5} - 1$ und $\lambda_6 = 1$. Unter anderem bestätigen die Autoren im folgenden eine Vermutung von K. Doliwka und M. Lassak, welche besagt, dass jedes konvexe Sechseck eine Seite der relativen Länge kleiner oder gleich $8 - 4\sqrt{3} = 1,071\ldots$ besitzt.

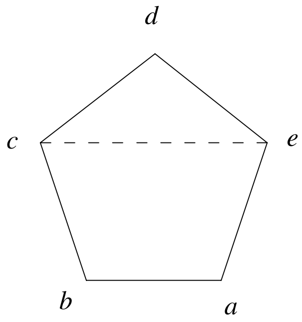

Fig. 1

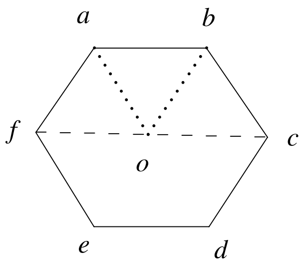

Fig. 2

In [1] Doliwka and Lassak showed that every convex pentagon (or quadrangle) has a relatively short side and a relatively long side. They conjectured that *every convex hexagon has a side of relative length at most* $8 - 4\sqrt{3} = 1.071\ldots$. We prove that this is true.

First, we give a hexagon which does not have any relatively short side. Let $\mathcal{H} = abcdef$ be a hexagon, where $\triangle bdf$ is a regular triangle, $|ab| = |bc| = |cd| = |de| = |ef| = |fa| = 1$, and $|ad| = |be| = |cf| = |bd|$ (see Fig. 3). It is easy to show that $ab \perp bc$. Draw $fm \perp bc$. Obviously, $|a'b'| = |fm| = \frac{1}{2}\tan(\frac{5\pi}{12}) = \frac{2+\sqrt{3}}{2}$, and we obtain $\lambda_{\mathcal{H}}(ab) = \frac{4}{2+\sqrt{3}} = 8 - 4\sqrt{3}$. In this way, we obtain that $\lambda_{\mathcal{H}}(ab) = \lambda_{\mathcal{H}}(bc) = \lambda_{\mathcal{H}}(cd) = \lambda_{\mathcal{H}}(de) = \lambda_{\mathcal{H}}(ef) = \lambda_{\mathcal{H}}(fa) = 8 - 4\sqrt{3} = 1.071\ldots > 1 = \lambda_6$. So, as a matter of fact, each side of the hexagon is relatively long.

Theorem 1. *Every convex hexagon has a side of relative length at most* $8 - 4\sqrt{3} = 1.071\ldots$*, and this upper bound is tight.*

Let $\mathcal{H}$ be a convex hexagon with vertices a, c', b, a', c, b'. For every non-degenerate affine transformation τ and for arbitrary points p, $q \in \mathcal{C}$, we know that $\lambda_{\mathcal{C}}(pq) = \lambda_{\tau(\mathcal{C})}(\tau(p)\tau(q))$. Thus, without loss of generality, we may assume that three non-adjacent vertices of the convex hexagon $\mathcal{H}$ form a regular triangle $\triangle abc$.

Let the center of $\triangle abc$ be o, and denote by $\overline{ao}$ the straight line passing through a, o. Similarly, we define straight lines $\overline{bo}$, $\overline{co}$. A convex hexagon $\mathcal{H} = ac'ba'cb'$ is called

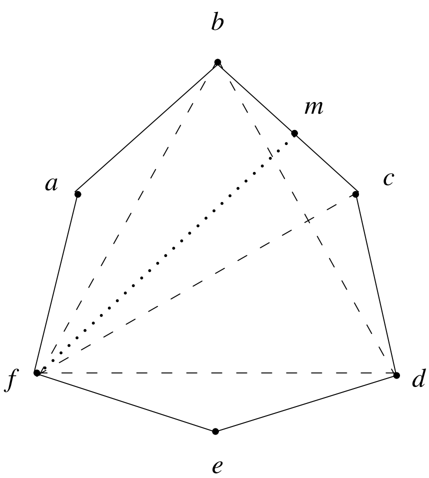

Fig. 3

a *special-regular hexagon*, if $\triangle abc$ is a regular triangle and $|aa'| = |bb'| = |cc'|$ with $a' \in \overline{ao}$, $b' \in \overline{bo}$, $c' \in \overline{co}$ (see Fig. 4).

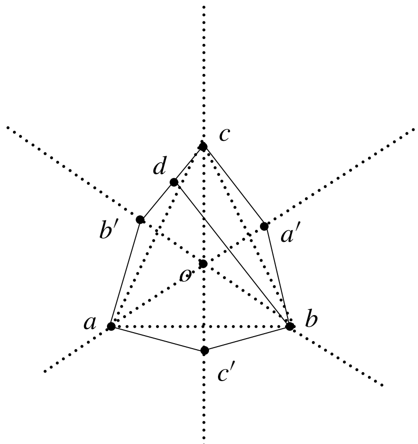

Fig. 4

Lemma 1. *The relative length of each side of a special-regular hexagon* $\mathcal{H} = ac'ba'cb'$ *is at most* $8 - 4\sqrt{3}$.

Proof. Without loss of generality, let $a = (-1, 0)$, $b = (1, 0)$, $c = (0, \sqrt{3})$, and $|aa'| = t > \sqrt{3}$. Then,

$$a' = \left(\frac{\sqrt{3}t}{2} - 1, \frac{t}{2}\right), \quad b' = \left(1 - \frac{\sqrt{3}t}{2}, \frac{t}{2}\right).$$

Take a point $d \in cb'$ such that the segments $a'c$ and bd are parallel (see Fig. 4). We then easily compute

$$d = \Big(\frac{2t - 2\sqrt{3}}{t - 2\sqrt{3}}, \frac{-t}{2 - \sqrt{3}t}\Big),$$

which leads to

$$|a'c|^2 = t^2 - 2\sqrt{3}t + 4, \quad |bd|^2 = t^2\Big(\frac{1}{(t - 2\sqrt{3})^2} + \frac{1}{(2 - \sqrt{3}t)^2}\Big).$$

Hence, we find

$$\lambda_{\mathcal{H}}(a'c) = \frac{2|a'c|}{|bd|} = \frac{-\sqrt{3}t^2 + 8t - 4\sqrt{3}}{t} = 8 - \sqrt{3}\Big(t + \frac{4}{t}\Big) \le 8 - 4\sqrt{3}.$$

Similarly, we can compute the relative length for each side of the hexagon and Lemma 1 is proved. □

Remark 1. When $t = 2$, that is, $|aa'| = |ab|$, we get the hexagon in Fig. 3, and the upper bound $8 - 4\sqrt{3}$ is attained. Generally speaking, when $\sqrt{3} < t < \frac{4\sqrt{3}}{3}$, we have $8 - \sqrt{3}(t + \frac{4}{t}) > 1$, so when $\sqrt{3} < t < \frac{4\sqrt{3}}{3}$, each side of the hexagon is relatively long.

Lemma 2. *If $\triangle abc$ is a regular triangle, $a' \in \overline{ao}$, $b' \in \overline{bo}$, $c' \in \overline{co}$ (see Fig.* 5*), then the convex hexagon $\mathcal{H} = ac'ba'cb'$ has a side of relative length at most* $8 - 4\sqrt{3}$.

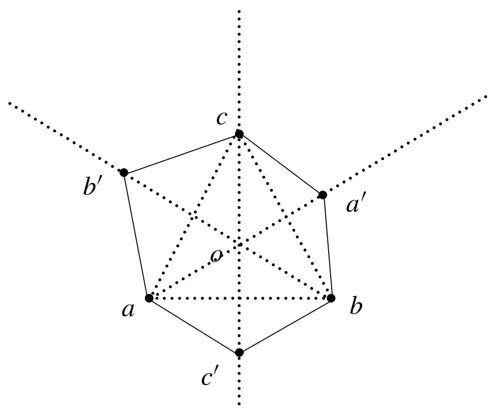

Fig. 5

Proof. Consider the segments aa', bb', and cc'. If $|aa'| = |bb'| = |cc'|$, then $\mathcal{H}$ is a special-regular hexagon, and we reach the conclusion by Lemma 1. Otherwise, we may assume that $|aa'| = \min\{|aa'|, |bb'|, |cc'|\}$. Then there exist points $b'' \in bb'$ and $c'' \in cc'$ such that $|aa'| = |bb''| = |cc''|$, and hence $\mathcal{H}_1 = ac''ba'cb''$ is a special-regular hexagon contained in hexagon $\mathcal{H} = ac'ba'cb'$. Therefore, $\lambda_{\mathcal{H}}(a'c) \le \lambda_{\mathcal{H}_1}(a'c) = 8 - 4\sqrt{3}$. Lemma 2 is proved. □

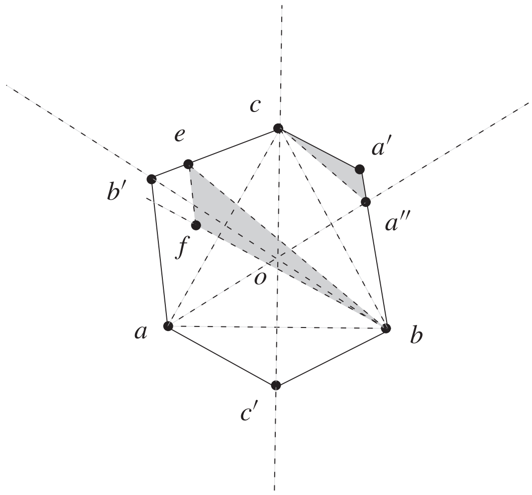

Fig. 6

Lemma 3. *If $\triangle abc$ is a regular triangle, $a' \notin \overline{ao}$, $b' \in \overline{bo}$, $c' \in \overline{co}$ (see Fig. 6), then the convex hexagon $\mathcal{H} = ac'ba'cb'$ has a side of relative length at most* $8 - 4\sqrt{3}$.

Proof. Denote by a'' the intersecting point of the segments ba' and $\overline{ao}$. Consider the convex hexagon $\mathcal{H}_1 = ac'ba''cb'$. Obviously, the hexagon $\mathcal{H}_1 \subseteq \mathcal{H} = ac'ba'cb'$. If $|bb'|$ or $|cc'|$ equals $\min\{|aa''|, |bb'|, |cc'|\}$, then, according to Lemma 2, $\lambda_{\mathcal{H}}(ab')$ or $\lambda_{\mathcal{H}}(ac')$ is at most $8 - 4\sqrt{3}$. If $|aa''|$ equals $\min\{|aa''|, |bb'|, |cc'|\}$, then, by Lemma 2, $|ca''|$ is at most $8 - 4\sqrt{3}$. Without loss of generality, we may assume that $|aa'| = |bb'| = |aa''|$, see Fig. 6. Then there exist points e and f such that $be \parallel a''c$, $bf \parallel a'c$, and $ef \parallel a'a''$. Then, $\triangle ca'a'' \sim \triangle bfe$, hence, $\frac{2|a''c|}{|be|} = \frac{2|a'c|}{|bf|} \leq 8 - 4\sqrt{3}$. Since $|bf|$ is smaller than the affine diameter parallel to $a'c$, therefore, $\lambda_{\mathcal{H}}(a'c) \leq 8 - 4\sqrt{3}$. The proof is complete. □

Remark 2. For the case $a' \in \overline{ao}$, $b' \notin \overline{bo}$, $c' \in \overline{co}$, or the case $a' \in \overline{ao}$, $b' \in \overline{bo}$, $c' \notin \overline{co}$ the conclusion of Lemma 3 can be reached similarly.

Lemma 4. *If $\triangle abc$ is a regular triangle, $a' \notin \overline{ao}$, $b' \notin \overline{bo}$, $c' \in \overline{co}$ (see Fig. 7), then the convex hexagon $\mathcal{H} = ac'ba'cb'$ has a side of relative length at most* $8 - 4\sqrt{3}$.

Proof. Denote by a'' the intersecting point of ba' and $\overline{ao}$, and b'' the intersecting point of cb' and $\overline{bo}$. If the hexagon $\mathcal{H}_1 = ac'ba''cb''$ is a special-regular hexagon, then the points a and c' are distant in relative length by at most $8 - 4\sqrt{3}$. Otherwise, we have three cases to consider. When $|cc'| = \min\{|aa''|, |bb''|, |cc'|\}$, by Lemma 2 we obtain $\lambda_{\mathcal{H}}(ac') \leq 8 - 4\sqrt{3}$; when $|aa''| = \min\{|aa''|, |bb''|, |cc'|\}$, then by Lemma 2 we have $\lambda_{\mathcal{H}_1}(ca'') \leq 8 - 4\sqrt{3}$, and so $\lambda_{\mathcal{H}}(ca') \leq 8 - 4\sqrt{3}$; when $|bb''| = \min\{|aa''|, |bb''|, |cc'|\}$, then by Lemma 2 we have $\lambda_{\mathcal{H}_1}(ab'') \leq 8 - 4\sqrt{3}$, and so $\lambda_{\mathcal{H}}(ab') \leq 8 - 4\sqrt{3}$. The proof is complete. □

Remark 3. For the case $a' \notin \overline{ao}$, $b' \in \overline{bo}$, $c' \notin \overline{co}$, or the case $a' \in \overline{ao}$, $b' \notin \overline{bo}$, $c' \notin \overline{co}$ the conclusion of Lemma 4 can be reached similarly.

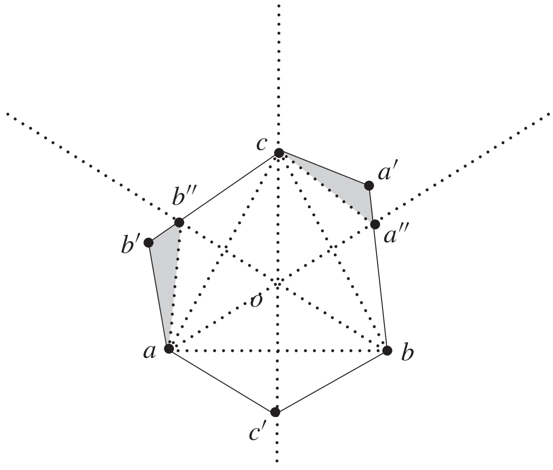

Fig. 7

Lemma 5. *If* $\triangle abc$ *is a regular triangle,* $a' \notin \overline{ao}$, $b' \notin \overline{bo}$, $c' \notin \overline{co}$ (*see Fig.* 8)*, then the convex hexagon* $\mathcal{H} = ac'ba'cb'$ *has a side of relative length at most* $8 - 4\sqrt{3}$.

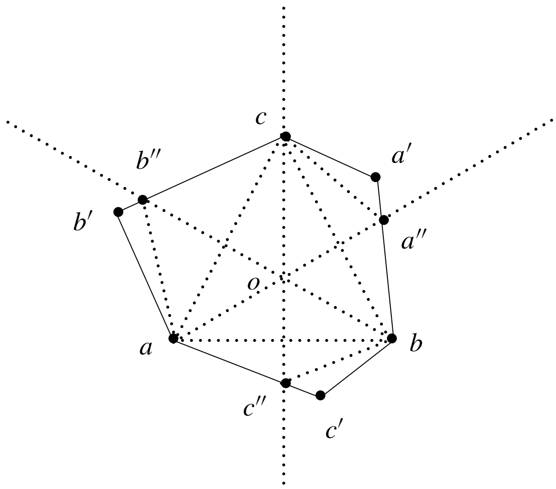

Fig. 8

Proof. Denote by a'' the intersecting point of ba' and $\overline{ao}$, b'' the intersecting point of cb' and $\overline{bo}$, and c'' the intersecting point of ac' and $\overline{co}$. Consider the hexagon $\mathcal{H}_1 = ac''ba''cb''$. Without loss of generality we may assume that $|aa''| = \min\{|aa''|, |bb''|, |cc''|\}$, then by Lemma 2 we have $\lambda_{\mathcal{H}_1}(ca'') \leq 8 - 4\sqrt{3}$ and hence $\lambda_{\mathcal{H}}(ca') \leq 8 - 4\sqrt{3}$. The proof is complete. □

Proof of Theorem 1. Combining Lemmas 1–5 we obtain Theorem 1, that is, the conjecture of Doliwka and Lassak is true. By Remark 1 the upper bound is tight. □

From Remark 1 we know that many convex hexagons have relative long sides, but not every convex polygon has a relative long side. For example we have the following result:

Theorem 2. *There exists a convex* 12*-gon which does not have any relatively long side.*

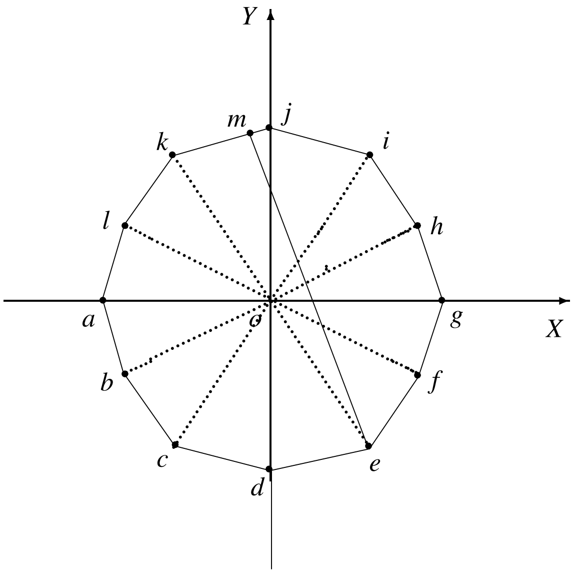

Fig. 9

Proof. We consider the convex 12-gon $\mathcal{Q} = abcdefghijkl$, as shown in Fig. 9, where $|oa| = |oc| = |oe| = |og| = |oi| = |ok| = t$ with $1 \leq t < \frac{2\sqrt{3}}{3}$, $|ob| = |od| = |of| = |oh| = |oj| = |ol| = 1$, the angle formed by any two consecutive segments with common point o equals $\frac{\pi}{6}$. By the symmetry of the construction of the convex 12-gon all sides of the 12-gon have the same relative length. We need only to compute one of them, say, $\lambda_{\mathcal{Q}}(gh)$. Obviously, $e = (\frac{t}{2}, -\frac{\sqrt{3}t}{2})$, $g = (t, 0)$, $h = (\frac{\sqrt{3}}{2}, \frac{1}{2})$, $j = (0, 1)$, $k = (-\frac{t}{2}, \frac{\sqrt{3}t}{2})$, and there exists a point $m \in jk$ such that the segments gh and em are parallel (see Fig. 9). The computation shows

$$|gh| = \sqrt{t^2 - \sqrt{3}t + 1}, \quad m = \Big(\frac{\sqrt{3}t^3 - \sqrt{3}t}{2\sqrt{3}t^2 - 8t + 2\sqrt{3}}, \frac{-3t^3 + 4\sqrt{3}t^2 - 5t}{2\sqrt{3}t^2 - 8t + 2\sqrt{3}}\Big),$$

hence,

$$|em| = \frac{2t\sqrt{t^2 - \sqrt{3}t + 1}}{|\sqrt{3}t^2 - 4t + \sqrt{3}|}.$$

Therefore, we obtain

$$\lambda_{\mathcal{Q}}(gh) = \frac{2|gh|}{|em|} = \frac{-\sqrt{3}t^2 + 4t - \sqrt{3}}{t} = 4 - \sqrt{3}\Big(t + \frac{1}{t}\Big) \le 2(2 - \sqrt{3}).$$

However, we can easily obtain that $\lambda_{12} = 2(2 - \sqrt{3})$ by setting $t = 1$ in the above equation since $\mathcal{Q}$ is a regular 12-gon when $t = 1$. Therefore, $\lambda_{\mathcal{Q}}(gh) \le \lambda_{12}$ and the proof is complete. □

Acknowledgement

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References

[1] Doliwka, K.; Lassak, M.: On Relatively Short and Long Sides of Convex Pentagons. *Geom. Dedicata* 56 (1995), 221–224.

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