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A simple constructive proof of Kronecker's Density Theorem

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Leopold Kronecker (1823–1891) achieved fame for his work in a variety of areas of mathematics, and notoriety for his unrelenting advocacy of a constructivist, almost finitist, philosophy of mathematics.

In the present note we give a direct and elementary proof of *Kronecker's Density Theorem* [2] (see also [3, pp. 49–109]):

Theorem 1. *If the real number θ is distinct from each rational multiple of π, then the set $\left\{e^{in\theta} \mid n \in \mathbb{Z}\right\}$ is dense in the unit circle.*

It is hard to believe that our proof is absolutely original, but it seems sufficiently natural to be worth presenting to the public.

Am Anfang der später von Hermann Weyl, Edmund Hlawka und deren Nachfolgern perfektionierten Theorie der Gleichverteilung stand unter anderem der Kroneckersche Dichtheitssatz: Hat eine komplexe Zahl vom Betrag 1 als Argument ein irrationales Vielfaches der Kreiszahl π, so bilden die Potenzen dieser komplexen Zahl eine dichte Teilmenge des Einheitskreises. Bekanntlich ist die Menge jener Potenzen endlich, wenn das Argument der Basis ein rationales Vielfaches von π ist. In der vorliegenden Arbeit geben die Autoren einen direkten und elementaren Beweis des Kroneckerschen Satzes an, auch um dessen konstruktive Gültigkeit nachzuweisen.

Moreover, in the spirit of Kronecker's views and work, it is one hundred per cent constructive.[1]

Proof. Since the set $\mathbb{Q}\pi$ of rational multiples of π is dense in $\mathbb{R}$, it is enough to prove that for each $t \in \mathbb{R}$ and each positive ε in $\mathbb{Q}\pi$ there exists $n \in \mathbb{Z}$ such that $\left|e^{it} - e^{in\theta}\right| < \varepsilon$, or, equivalently, that for all such t, ε there exist integers p, q such that $|p\theta - t + 2q\pi| < \varepsilon$. We may assume that $0 < \theta < 2\pi$: for in the general case, since θ is distinct from each rational multiple of π, there exists an integer k such that $0 < \theta - 2k\pi < 2\pi$; if we can compute $p, q \in \mathbb{Z}$ such that

$$|p(\theta - 2k\pi) - t + 2q\pi| < \varepsilon,$$

then we have

$$|p\theta - t + 2(q - kp)\pi| < \varepsilon.$$

Next we show that it suffices to prove the case $t = 0$. Indeed, supposing that we have found $p, q \in \mathbb{Z}$ such that $|p\theta + 2q\pi| < \varepsilon$, for arbitrary $t \in \mathbb{R}$ we can find an integer k such that

$$\left|k - \frac{t}{p\theta + 2q\pi}\right| < 1.$$

(Note that $p\theta + 2q\pi$ is distinct from 0, by our hypothesis on θ). Then

$$|kp\theta - t + 2kq\pi| = |p\theta + 2q\pi| \cdot \left|k - \frac{t}{p\theta + 2q\pi}\right| < |p\theta + 2q\pi| < \varepsilon.$$

For our final simplification we may assume that $0 < \varepsilon < \min\left\{\frac{\pi}{2}, \theta\right\}$.

The idea behind the rest of the proof is simple. Starting at the point $e^{i\theta}$, we move anticlockwise round the unit circle in steps of arc length θ until we pass the positive x-axis. Since θ is not a rational multiple of π, this brings us to a point $e^{i\theta_1}$ with $0 < \theta_1 < \theta$, where $2\pi + \theta_1$ is an integer multiple of θ; so $\theta_1 - \theta$ is distinct from $-\varepsilon$. If $\theta_1 - \theta > -\varepsilon$, then we are finished; if $\theta_1 - \theta < -\varepsilon$, then we repeat the procedure with θ replaced by θ_1. It is easy to give an upper bound for the number of times this procedure must be iterated to ensure that we arrive at the desired approximation to 0.

Here is the precise argument. Taking $\theta_0 = \theta$, suppose that for some $k \geqslant 1$ we have found real numbers $\theta_0 = \theta, \ldots, \theta_{k-1}$, positive and distinct from each rational multiple of π, and integers $p_1, \ldots, p_{k-1} > 1$ such that

$$0 < \theta_i = p_i\theta_{i-1} - 2\pi < \theta_{i-1} \quad (1 \leq i \leq k-1). \tag{1}$$

Compute

$$p_k = \min\{n \in \mathbb{N} \mid n\theta_{k-1} > 2\pi\},$$

[1] Several other proofs of Kronecker's Density Theorem, including Kronecker's original one, are given in Chapter XXIII of [1]. – From today's perspective, Kronecker's theorem stood at the beginning of the theory of uniform distribution continued by Hermann Weyl, and brought to some perfection by the Vienna school around Edmund Hlawka. Kronecker's theorem is a by-product of Weyl's approach to uniform distribution, which can be constructivised relatively easily [4].

and set $\theta_k = p_k\theta_{k-1} - 2\pi$. Then $p_k > 1$, $\theta_k > 0$, and

$$\theta_k - \theta_{k-1} = (p_k - 1)\,\theta_{k-1} - 2\pi < 0\,.$$

Also, θ_k and $\theta_k - \theta_{k-1}$ are distinct from each rational multiple of π, so either $\theta_k - \theta_{k-1} > -\varepsilon$ or else $\theta_k - \theta_{k-1} < -\varepsilon$ (recall that $\varepsilon \in \mathbb{Q}\pi$). In the first case, a simple induction using (1) shows that there exists an integer q such that

$$0 < \theta_k - \theta_{k-1} = (p_k - 1)\,\theta_{k-1} - 2\pi = (p_k - 1)\,p_{k-1}\cdots p_1\theta - 2q\pi < \varepsilon,$$

and we terminate the procedure. In the second case we proceed to the construction of p_{k+1} and θ_{k+1}. To show that this process must eventually stop, choose a positive integer M such that $M\varepsilon > \theta$. If the process did not end with the construction of θ_M and p_M, then for $1 \le k \le M$ we would have $0 < \theta_k < \theta_{k-1} - \varepsilon$; whence $0 < \theta_M < \theta - M\varepsilon < 0$, a contradiction. Thus the process must stop at or before the Mth iteration. □

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