Elem. Math. 62 (2007) 102 – 105
0013-6018/07/030102-4

Elemente der Mathematik

Affine congruence by dissection of intervals

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1 Introduction and results

Tarski's circle squaring problem (see [8]) has motivated the following question: Can a circular disc be dissected into finitely many topological discs such that images of these pieces under suitable Euclidean motions form a dissection of a square? Dubins, Hirsch, and Karush give a negative answer in [1]. However, one can get positive results if the group of Euclidean motions is replaced by suitable other groups of affine maps of the plane (see [3, 5, 6, 7]). The general concept behind these phenomena is the congruence by dissection of discs with respect to some fixed group of affine transformations of $\mathbb{R}^2$.

Let d denote the Euclidean distance in the plane $\mathbb{R}^2$. We recall that a topological disc D is the image of the closed unit disc $\{x \in \mathbb{R}^2 : d(x, 0) \leq 1\}$ under a homeomorphism of the plane onto itself. We say that D is *dissected* into the discs $D_1, \ldots, D_n$ if $D = D_1 \cup \ldots \cup D_n$ and $\operatorname{int}(D_i) \cap \operatorname{int}(D_j) = \emptyset$ for $1 \leq i < j \leq n$, $\operatorname{int}(D_i)$ denoting the interior of D_i.

Given a group $\mathcal{G}$ of affine transformations of $\mathbb{R}^2$, two topological discs D, E are called *congruent by dissection with respect to* $\mathcal{G}$ if and only if there exist dissections of D and E into the same finite number $n \geq 1$ of subdiscs $D_1, \ldots, D_n$ and $E_1, \ldots, E_n$, respectively,

In der vorliegenden Arbeit wird der Begriff der Zerlegungsgleichheit von Intervallen auf der reellen Zahlengeraden bezüglich einer Gruppe $\mathcal{G}$ affiner Transformationen eingeführt und studiert. Dabei werden zwei kompakte Intervalle $I, J \subseteq \mathbb{R}$ zerlegungsgleich bezüglich $\mathcal{G}$ genannt, wenn I eine Zerlegung in endlich viele Teilintervalle besitzt, welche durch Transformationen aus $\mathcal{G}$ in eine Zerlegung von J überführt werden können. Es zeigt sich, dass $\mathcal{G}$ genau dann die Zerlegungsgleichheit beliebiger Intervalle positiver Länge erlaubt, wenn $\mathcal{G}$ alle Translationen und eine Kontraktion besitzt. Dagegen ermöglicht $\mathcal{G}$ die Zerlegungsgleichheit beliebiger Intervalle gleicher Länge dann und nur dann, wenn alle Translationen zu $\mathcal{G}$ gehören.

such that, for $1 \leq i \leq n$, D_i and E_i are congruent with respect to $\mathcal{G}$ (that is, there exists $\varphi_i \in \mathcal{G}$ such that $E_i = \varphi_i(D_i)$). In this case we say that the congruence by dissection of D and E is realized by *n pieces of dissection*, namely by $D_1, \dots, D_n$ or $E_1, \dots, E_n$, respectively.

Theorem 2 from [7] characterizes the groups $\mathcal{G}$ that admit a congruence by dissection of any two topological discs. It says in particular the following:

Theorem 0. *Let $\mathcal{G}$ be a group of affine transformations of $\mathbb{R}^2$. Then the following are equivalent:*

(a) *Any two topological discs $D, E \subseteq \mathbb{R}^2$ are congruent by dissection with respect to $\mathcal{G}$.*

(b) *Any two topological discs $D, E \subseteq \mathbb{R}^2$ of the same two-dimensional Hausdorff measure are congruent by dissection with respect to $\mathcal{G}$.*

(c) *$\mathcal{G}$ contains a contraction and every orbit $\mathcal{G}(x)$, $x \in \mathbb{R}^2$, is dense in $\mathbb{R}^2$.*

Here a map $\varphi \in \mathcal{G}$ is called a contraction if there is a constant $0 < c < 1$ such that $d(\varphi(x_1), \varphi(x_2)) \leq c\, d(x_1, x_2)$ for all $x_1, x_2 \in \mathbb{R}^2$. The orbit $\mathcal{G}(x)$ is defined by $\mathcal{G}(x) = \{\varphi(x) : \varphi \in \mathcal{G}\}$.

In the present note we ask for a similar characterization of groups $\mathcal{G}$ in the one-dimensional case. Then compact intervals of positive length are the analogues of topological discs. With this replacement, congruence by dissection can be defined as above. Since in the one-dimensional case the family of the compact intervals coincides with that of all connected polyhedra, the following results can be understood as contributions to the theory of congruence by dissection of polyhedra with polyhedral pieces of dissection, too (see Chapters 1 and 2 of [2]).

It turns out that the one-dimensional versions of (a) and (b) are not equivalent. We obtain the following two characterizations:

Theorem 1. *Let $\mathcal{G}$ be a group of affine transformations of $\mathbb{R}$. Then the following are equivalent:*

(i) *Any two compact intervals $I, J \subseteq \mathbb{R}$ of positive length are congruent by dissection with respect to $\mathcal{G}$.*

(ii) *Any two compact intervals $I, J \subseteq \mathbb{R}$ of positive length admit a congruence by dissection with respect to $\mathcal{G}$ that uses only two pieces of dissection.*

(iii) *$\mathcal{G}$ contains a contraction and acts transitively on $\mathbb{R}$.*

(iv) *$\mathcal{G}$ contains a contraction and all translations.*

Theorem 2. *Let $\mathcal{G}$ be a group of affine transformations of $\mathbb{R}$. Then the following are equivalent:*

(i)′ *Any two compact intervals $I, J \subseteq \mathbb{R}$ of the same positive length are congruent by dissection with respect to $\mathcal{G}$.*

(ii)′ *Any two compact intervals $I, J \subseteq \mathbb{R}$ of the same positive length are congruent with respect to $\mathcal{G}$.*

(iii)′ *$\mathcal{G}$ acts transitively on $\mathbb{R}$.*

(iv)′ *$\mathcal{G}$ contains all translations.*

Note that the one-dimensional analogue of (c) does not imply (iii) and not even (iii)′. This contradicts the first impression that congruence by dissection of intervals should be much easier realizable than that of discs, because it is much more elementary. The larger flexibility of congruence by dissection of discs rests on the huge freedom concerning the possible shapes of pieces of dissection.

In contrast with (i) and (i)′, the weaker condition of the existence of a congruence by dissection of any two compact intervals $I, J \subseteq \mathbb{R}$ of a *fixed* positive length does not imply the transitivity of $\mathcal{G}$. For example, the group $\mathcal{G} = \mathbb{Z}$ of integer translations gives a congruence by dissection of any two intervals $I = [a, a+1]$ and $J = [b, b+1]$ of length one. Indeed, we find $l \in \mathbb{Z}$ such that $a \leq b + l < a + 1$ and obtain trivially $J = I - l$ if $a = b + l$. Otherwise we have $I = [a, b+l] \cup [b+l, a+1]$ and $([b+l, a+1] - l) \cup ([a, b+l] + 1 - l) = [b, a+1-l] \cup [a+1-l, b+1] = J$.

2 Proofs

The proofs of Theorems 1 and 2 are presented simultaneously.

1. (iii)⇒(iv) and (iii)′ ⇒(iv)′. Let $\mathcal{G}$ be transitive. We have to show that, for every $x_0 \in \mathbb{R}$, there is a translation in $\mathcal{G}$ mapping 0 onto x_0. Let $\varphi \in \mathcal{G}$ be such that $\varphi(0) = x_0$. If φ is a translation we are done. Otherwise φ has a fixed point x_1. We pick $\psi \in \mathcal{G}$ with $\psi(x_0) = x_1$. Then $\psi^{-1}\varphi^{-1}\psi\varphi$ is a translation that maps 0 onto x_0.

2. (iv)′ ⇒(ii)′ is obvious. For proving (iv)⇒(ii) we suppose (iv) to be satisfied and consider two fixed intervals I, J of length $a, b > 0$, respectively. Say $a < b$, because the case $a = b$ is trivial. By (iv), there is a constant $c > 1$ such that $\mathcal{G}$ contains all dilatations with factor c^k, $k \in \mathbb{Z}$. We pick $k_0 \geq 1$ such that $a\,c^{k_0} > b$ and dissect I into subintervals I_1, I_2 of length $a_1 = \frac{b-a}{c^{k_0}-1}$, $a_2 = a - \frac{b-a}{c^{k_0}-1}$, respectively. Then J can be dissected into images $\varphi_1(I_1)$ and $\varphi_2(I_2)$, since $b = c^{k_0}\,a_1 + a_2$. This yields (ii). (In the context of affine congruence by dissection of polyhedra the proof of (iv)⇒(ii) was already given in [4].)

3. (ii)⇒(i) and (ii)′ ⇒(i)′ are trivial.

4. (i)⇒(iii) and (i)′ ⇒(iii)′. If $\mathcal{G}$ admits a congruence by dissection of intervals of different length then $\mathcal{G}$ clearly must contain a contraction. We prepare the proof of the remaining implication (i)′ ⇒(iii)′ by a lemma.

Lemma. *Let two intervals $I = [a_1, a_2]$ and $J = [b_1, b_2]$ be congruent by dissection with respect to a group $\mathcal{G}$ of affine transformations of $\mathbb{R}$. Then*

$$\mathcal{G}(a_1) \cap \big(\mathcal{G}(a_2) \cup \mathcal{G}(b_1) \cup \mathcal{G}(b_2)\big) \neq \emptyset \quad \textit{and} \quad \mathcal{G}(b_2) \cap \big(\mathcal{G}(a_1) \cup \mathcal{G}(a_2) \cup \mathcal{G}(b_1)\big) \neq \emptyset.$$

The proof even comprises arbitrary groups $\mathcal{G}$ of homeomorphisms of $\mathbb{R}$. The present version goes back to an anonymous hint.

Proof. According to the supposition there exist a dissection $I = I_1 \cup \ldots \cup I_n$ into subintervals $I_i = [x_{i-1}, x_i]$ with $a_1 = x_0 < x_1 < \ldots < x_n = a_2$, a dissection $J = J_1 \cup \ldots \cup J_n$ into subintervals $J_i = [y_{i-1}, y_i]$ with $b_1 = y_0 < y_1 < \ldots < y_n = b_2$, maps $\varphi_i \in \mathcal{G}$, and a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\varphi_i(I_i) = J_{\pi(i)}$, $1 \leq i \leq n$.

Let Γ be a bipartite graph between $A = \{(x_i, 1) : 0 \le i \le n\}$ and $B = \{(y_i, 2) : 0 \le i \le n\}$ with the following edges: $(x_{i-1}, 1)$ (considered as the left-hand end-point of I_i) is connected with $(\varphi_i(x_{i-1}), 2)$ and $(x_i, 1)$ (representing the right-hand end-point of I_i) is connected with $(\varphi_i(x_i), 2)$, $1 \le i \le n$. If $\varphi_i(x_i) = \varphi_{i+1}(x_i)$ then $(x_i, 1)$ and $(\varphi_i(x_i), 2)$ are connected by two edges.

Obviously, the vertices $(x_0, 1)$, $(x_n, 1)$, $(y_0, 2)$, and $(y_n, 2)$ are of degree 1, whereas all other vertices have degree 2. Hence the connected component of Γ starting at $(x_0, 1) = (a_1, 1)$ is a path whose other end-point is one of $(x_n, 1) = (a_2, 1)$, $(y_0, 2) = (b_1, 2)$, and $(y_n, 2) = (b_2, 2)$. This yields $\mathcal{G}(a_1) \cap \{a_2, b_1, b_2\} \ne \emptyset$. The same argument gives $\mathcal{G}(b_2) \cap \{a_1, a_2, b_1\} \ne \emptyset$. □

We come back to the proof of (i)$'$ $\Rightarrow$(iii)$'$. It is to show that (i)$'$ implies $\mathcal{G}(a) \cap \mathcal{G}(b) \ne \emptyset$ for all $a, b \in \mathbb{R}$, $a < b$. By (i)$'$, the intervals $I = \left[a, \frac{a+b}{2}\right]$ and $J = \left[\frac{a+b}{2}, b\right]$ are congruent by dissection with respect to $\mathcal{G}$. If $\mathcal{G}\left(\frac{a+b}{2}\right) \cap \mathcal{G}(b) \ne \emptyset$ then we have $\mathcal{G}\left(\frac{a+b}{2}\right) = \mathcal{G}(b)$ and the first part of the lemma yields the claim $\mathcal{G}(a) \cap \mathcal{G}(b) \ne \emptyset$. In the opposite case $\mathcal{G}\left(\frac{a+b}{2}\right) \cap \mathcal{G}(b) = \emptyset$ we obtain $\mathcal{G}(b) \cap \mathcal{G}(a) \ne \emptyset$ by the second part of the lemma. So (i)$'$ $\Rightarrow$(iii)$'$ is verified and the proofs of Theorems 1 and 2 are complete. □

References

[1] Dubins, L.; Hirsch, M.W.; Karush, J.: Scissor congruence. *Israel J. Math.* 1 (1963), 239–247.

[2] Hadwiger, H.: *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer-Verlag, Berlin-Göttingen-Heidelberg 1957.

[3] Hertel, E.; Richter, C.: Squaring the circle by dissection. *Beiträge Algebra Geom.* 44 (2003), 47–55.

[4] Meier, C.: Zerlegungsähnlichkeit von Polyedern. *J. Reine Angew. Math.* 253 (1972), 193–202.

[5] Richter, C.: Congruence by dissection of topological discs – An elementary approach to Tarski's circle squaring problem. *Discrete Comput. Geom.* 28 (2002), 427–442.

[6] Richter, C.: The minimal number of pieces realizing affine congruence by dissection of topological discs. *Period. Math. Hungar.* 46 (2003), 203–213.

[7] Richter, C.: Affine congruence by dissection of discs – appropriate groups and optimal dissections. *J. Geom.* 84 (2005), 117–132.

[8] Tarski, A.: Problème 38. *Fund. Math.* 7 (1925), 381.

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[1]This research was supported by DFG grant RI 1087/3. The paper was written during an extended stay of the author at the Mathematical Institute of the University Paris VI.