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Note from the Editorial Board
Note on rectangles with vertices on prescribed circles

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Inspired by [2], Ionascu and Stanica considered the following problem: The four subsequent vertices of a rectangle R in the plane are at distances r_1, r_2, r_3, r_4 from the origin. Given these data, what can be said about the area A of R? In a recent paper [1] they proved

Theorem 1. (a) *For given $r_i \geq 0$, rectangles R of the described kind exist iff*

$$r_1^2 + r_3^2 = r_2^2 + r_4^2 . \tag{1}$$

(b) *The areas of these rectangles lie between the bounds*

$$A_{\min} = |r_2 r_4 - r_1 r_3| , \qquad A_{\max} = r_2 r_4 + r_1 r_3 .$$

The authors call their problem "unusual because of its surprisingly simple answer in spite of our rather laborious solution" (which takes 10 pages). In this note we shall prove Theorem 1 in a simple way, making use of no more than the Pythagorean theorem and the formula for the derivative of a product.

After cyclic reordering, and neglecting special or degenerate cases, we may assume that $0 < r_1 < r_i$ $(2 \leq i \leq 4)$. We may also assume that the sides of the rectangle are parallel to the axes and that the point $P_1 = (x, y)$ on the circle of radius r_1 is the lower lefthand vertex of R (Fig. 1). From P_1 we draw a horizontal to the right and obtain the vertex $P_2 = (\bar{x}, y)$ on the circle of radius r_2; in a similar way, going from P_1 vertically upwards, we obtain the vertex $P_4 = (x, \bar{y})$ on the circle of radius r_4. So far we have

$$\begin{aligned} x^2 + y^2 &= r_1^2 , \\ \bar{x}^2 + y^2 &= r_2^2 , \qquad \bar{x} > 0 , \\ x^2 + \bar{y}^2 &= r_4^2 , \qquad \bar{y} > 0 . \end{aligned} \tag{2}$$

The upper righthand vertex of R is $P_3 = (\bar{x}, \bar{y})$, and it lies on the circle of radius r_3 iff

$$r_3^2 = \bar{x}^2 + \bar{y}^2 = r_2^2 - y^2 + r_4^2 - x^2 = r_2^2 + r_4^2 - r_1^2 ,$$

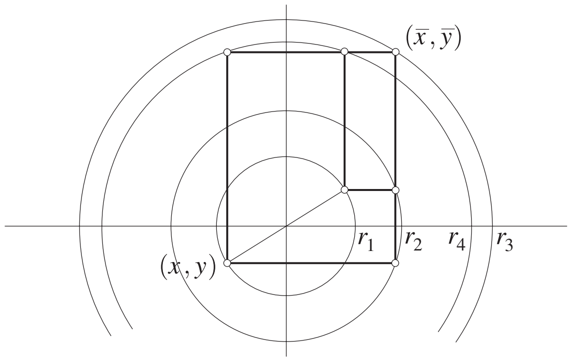

Figure 1

i.e., iff (1) holds. Since the above construction is always possible, part (a) of the theorem follows.

Remark. The midpoint of R bisects both diagonals. It follows that (1) can be seen as an instance of the so-called Pappus-Fagnano-Legendre formula $a^2 + b^2 = (c^2 + z^2)/2$ where $z/2$ is the length of the median through C of a triangle with sides a, b, c.

For our extremal problem we have to consider admissible variations dx, dy, $d\bar{x}$, $d\bar{y}$ of the quantities x, y, $\bar{x}$, $\bar{y}$. To this end we differentiate the equations (2) and obtain

$$x\,dx + y\,dy = 0\,, \quad \bar{x}\,d\bar{x} + y\,dy = 0\,, \quad x\,dx + \bar{y}\,d\bar{y} = 0 \tag{3}$$

from which we easily infer

$$d\bar{x} = \frac{x}{\bar{x}}\,dx\,, \qquad d\bar{y} = \frac{y}{\bar{y}}\,dy\,. \tag{4}$$

The area of R is $A = (\bar{x} - x)(\bar{y} - y)$; it is maximal or minimal iff

$$dA = (d\bar{x} - dx)(\bar{y} - y) + (\bar{x} - x)(d\bar{y} - dy) = 0\,.$$

Using (4) we get

$$dA = \frac{x - \bar{x}}{\bar{x}}(\bar{y} - y)\,dx + (\bar{x} - x)\frac{y - \bar{y}}{\bar{y}}dy = -\frac{A}{\bar{x}\,\bar{y}}(\bar{y}\,dx + \bar{x}\,dy)\,.$$

So the condition $dA = 0$ reduces to $\bar{y}\,dx + \bar{x}\,dy = 0$. Combined with the first equation (3) this shows that in the stationary situation we necessarily have

$$\frac{y}{x} = \frac{\bar{x}}{\bar{y}}\,. \tag{5}$$

In order to determine the maximal and minimal areas of the rectangle we argue as follows: From (5) and (2) we get

$$\frac{\bar{x}^2}{\bar{y}^2} = \frac{y^2}{x^2} = \frac{r_2^2 - \bar{x}^2}{r_4^2 - \bar{y}^2}\,, \qquad \text{whence} \qquad \bar{x}^2(r_4^2 - \bar{y}^2) = \bar{y}^2(r_2^2 - \bar{x}^2)\,,$$

and after cancelling terms we see that

$$\frac{\bar{y}}{\bar{x}} = \frac{r_4}{r_2}, \qquad \frac{y}{x} = \frac{r_2}{r_4},$$

the latter using (5) again.

We now know the ratios of these coordinates as well as their Pythagorean sums, whence they must be

$$x = \pm\frac{r_4}{\sqrt{r_2^2 + r_4^2}}\, r_1, \quad y = \pm\frac{r_2}{\sqrt{r_2^2 + r_4^2}}\, r_1, \quad \bar{x} = \frac{r_2}{\sqrt{r_2^2 + r_4^2}}\, r_3, \quad \bar{y} = \frac{r_4}{\sqrt{r_2^2 + r_4^2}}\, r_3.$$

Inserting these values into the formula for A we obtain, using (1):

$$A = \frac{(r_2 r_3 \mp r_4 r_1)(r_4 r_3 \mp r_2 r_1)}{r_2^2 + r_4^2} = \frac{r_1^2 + r_3^2}{r_2^2 + r_4^2}\, r_2 r_4 \mp r_1 r_3 = r_2 r_4 \mp r_1 r_3.$$

This leads to

$$A_{\min} = r_2 r_4 - r_1 r_3, \qquad A_{\max} = r_2 r_4 + r_1 r_3,$$

as stated. It is easily seen that for the maximum the origin lies in the interior of R whereas for the minimum it is in the exterior.

References

[1] Ionascu, E.J.; Stanica, P.: Extremal values for the area of rectangles with vertices on concentrical circles. *Elem. Math.* 62 (2007), 30–39.

[2] Zahlreich Problems Group: Problem 11057. *Amer. Math. Monthly* 111 (2004), 64. Solution: ibid. 113 (2006), 82.

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