Elem. Math. 63 (2008) 6 – 13
0013-6018/08/010006-8

**Elemente der Mathematik**

# Thébault's theorem

Darko Veljan and Vladimir Volenec

Darko Veljan is professor of mathematics at the University of Zagreb. He obtained his Ph.D. in 1979 at Cornell University. His interests include algebraic topology, combinatorics, geometry and some aspects of mathematical education.

Vladimir Volenec is professor of mathematics at the University of Zagreb. He obtained his Ph.D. in 1971 at the University of Zagreb. His interests include non-associative algebraic structures and their applications in geometry.

## 1 Thébault's theorem

The following theorem is usually called *Thébault's theorem*. We refer to Fig. 1.

**Theorem 1.** *Let $(I, r)$ be the incircle of a triangle $\triangle ABC$ ($I$ is the center and $r$ is the radius), and $T$ any point on the line $BC$. Let $(P, r_1)$ and $(Q, r_2)$ be two circles touching the lines $AT$ and $BC$ and the circumcircle $ABC$. Then the three centers $P$, $Q$ and $I$ are collinear and the following relations hold:*

$$PI : IQ = \tau^2, \tag{1}$$

$$r_1 + r_2\tau^2 = r(1 + \tau^2), \tag{2}$$

*where $2\theta = \angle ATB$, and $\tau = \tan\theta$.*

Thébault's theorem was originally proposed in 1938 as a problem in the American Mathematical Monthly by the French geometer Victor Thébault [14]. Thébault's theorem remained an open problem for some 45 years, until the proof appeared in 1983 [13]. This

Ende der dreissiger Jahre des letzten Jahrhunderts stellte der französische Geometer Victor Thébault im *American Mathematical Monthly* eine Aufgabe zur Dreiecksgeometrie. Überraschenderweise wurde die erste Lösung dieses Problems erst knapp ein halbes Jahrhundert später veröffentlicht, ebenfalls im *Monthly*. Die Autoren liefern in diesem Beitrag einen weiteren, elementaren Beweis des Satzes von Thébault, indem sie eine Charakterisierung von Kreisen geben, die eine Dreiecksseite und den Umkreis des Dreiecks berühren. Darüber hinaus diskutieren sie am Ende mögliche Verallgemeinerungen des Satzes von Thébault in drei Dimensionen.

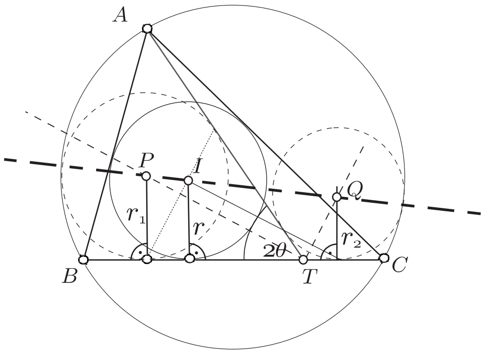


Fig. 1

proof used analytic geometry and involved lengthy computations. As it is often the case in situations like this one, a series of new, short and more elegant proofs appeared after that. So, for example, [15] and [5] use trigonometry, [12] and [7] are synthetic proofs, [10] uses computer algebra software for an (again analytic) proof etc. Some proofs actually showed a more general claim than Thébault's original theorem. But some proofs treated only special cases; e.g. [3] treated only the case when $AT$ is perpendicular to $BC$. Surprisingly, there is a short and nice solution of the original problem which was received back in 1975, but published only in 2003 [2], since (in Editorial comment's words) "...through circumstances lost in the mists of time, it somehow fell through the cracks." The solutions [4] and [17] referred to a proof [16] (in Dutch), which was prior to [2]. Let us note here that a wrong version of formula (2) appeared in the original [14], but was corrected in [2].

In our approach here, in proving Thébault's theorem we first give a necessary and sufficient condition that a circle touches one side and the circumcircle of a triangle (Theorem 2). We use this criterion to approve a geometric construction of "Thébault's circles" (Theorem 3), and then we give a short proof of Thébault's original theorem. Some easy consequences of our considerations are also discussed (Theorems 4 and 5).

Our results include all the results known to the authors that treat and generalize Thébault's original theorem. We shall make some comments later in the text.

Finally, based on some calculations, we note that more-or-less "obvious" space versions of Thébault's theorem do not hold. So, the question is what is the space analogue of Thébault's theorem, if there is any at all.

## 2 Auxiliary results

**Theorem 2.** *Suppose a circle $\mathcal{K} = (P, \rho)$ touches a line $BC$ at the point $U$ and let the points $A$ and $P$ be on the same side of this line. Then the circle $\mathcal{K}$ touches the circle $ABC$ from the inside if and only if for the oriented distances $BU$ and $UC$ (with $BC = a$) we have*

$$BU \cdot UC = a\rho \tan\frac{A}{2} = a\rho\alpha. \tag{3}$$

*Proof.* Let $(O, R)$ be the circumcircle of the triangle $ABC$, and let $L$ be the midpoint of $\overline{BC}$. Then (see Fig. 2) the oriented distance from $O$ to the line $BC$ is given by $LO = R\cos A$. Note that this distance can be positive, but also negative if $O$ is "below" the line $BC$. Since $UP = \rho$, we have the following equality

$$OP^2 = LU^2 + (\rho - R\cos A)^2. \tag{4}$$

Since $BU \cdot UC = (LU - LB)(LC - LU) = \left(LU + \frac{a}{2}\right)\left(\frac{a}{2} - LU\right) = \frac{a^2}{4} - LU^2 = R^2\sin^2 A - LU^2$, we obtain

$$LU^2 = R^2\sin^2 A - BU \cdot UC. \tag{5}$$

The circle $\mathcal{K}$ touches the circle $ABC$ from the inside if and only if $OP^2 = (R-\rho)^2$. From (4) and (5) it follows that this is equivalent to

$$R^2\sin^2 A - BU \cdot UC + (\rho - R\cos A)^2 = (R-\rho)^2,$$

or, by rearranging a bit,

$$BU \cdot UC = 2R\rho(1 - \cos A).$$

Since $2R(1-\cos A) = a\dfrac{1-\cos A}{\sin A} = a\tan\dfrac{A}{2}$, it follows that this is equivalent to the equality (3). $\square$

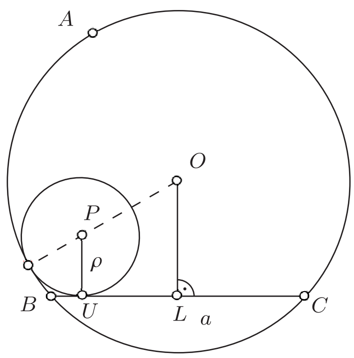


Fig. 2

Denote, as usual, by $p := (a+b+c)/2$ the half-perimeter of the triangle $\triangle ABC$, by $\triangle$ its area, by $r$ its inradius, and put $\alpha := \tan\dfrac{A}{2}$, $\beta := \tan\dfrac{B}{2}$, $\gamma := \tan\dfrac{C}{2}$.

Then, since $\alpha = \dfrac{r}{p-a}$, $\beta = \dfrac{r}{p-b}$, $\gamma = \dfrac{r}{p-c}$, it follows

$$\beta\gamma = \frac{r^2}{(p-b)(p-c)} = \frac{p-a}{p}, \tag{6}$$

and then, from $\alpha\beta\gamma = \dfrac{r}{p-a} \cdot \dfrac{p-a}{p} = \dfrac{r}{p}$, it follows

$$\alpha\beta\gamma = \frac{r^2}{\Delta}. \tag{7}$$

The following theorem may be viewed as a recipe for constructing (by ruler and compass) circles that touch a given circle from the inside and two given chords. We refer to Fig. 3.

**Theorem 3.** ([15], [4], [8]) *Let $(I, r)$ be the incircle of a triangle $\triangle ABC$, and $T$ any point on the line $BC$. Let the perpendiculars from $I$ to the bisectors of the angles $\angle ATB$ and $\angle ATC$ meet $BC$ at the points $U$ and $V$, and let the normals to $BC$ at $U$ and $V$ meet these bisectors at $P$ and $Q$, respectively. Then the circles with centers $P$ and $Q$ and radii $r_1 = PU$ and $r_2 = QV$ touch the lines $AT$ and $BC$ and the circle $ABC$ from the inside.*

*Proof.* Let $\tau = \tan\Theta$, where $2\Theta = \angle ATB$. Further, let $D$ be the foot of the perpendicular from $I$ to $BC$, and $ID = r$ be the inradius of $\triangle ABC$. Then $UD = r\tau$, $BD = r\cot\dfrac{B}{2} = \dfrac{r}{\beta}$, and also $DC = \dfrac{r}{\gamma}$. Hence, $BU = BD - UD = \dfrac{r}{\beta} - r\tau = \dfrac{r}{\beta}(1-\beta\tau)$, and $UC = UD + DC = r\tau + \dfrac{r}{\gamma} = \dfrac{r}{\gamma}(1+\gamma\tau)$.

From (7) we therefore infer

$$BU \cdot UC = \Delta\alpha(1-\beta\tau)(1+\gamma\tau). \tag{8}$$

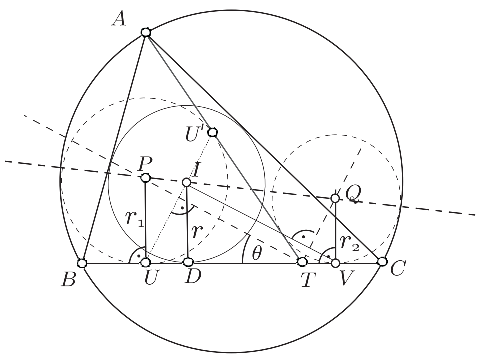


Fig. 3

Let $h_a$ be the height from the vertex $A$ of $\triangle ABC$. Then (see Fig. 3)

$$\begin{aligned} BT &= h_a(\cot B + \cot 2\Theta) = \frac{2\Delta}{a}\left(\frac{1-\beta^2}{2\beta} + \frac{1-\tau^2}{2\tau}\right) \\ &= \frac{rp}{a\beta\tau}(\tau - \beta^2\tau + \beta - \beta\tau^2) = \frac{rp}{a\beta\tau}(\tau+\beta)(1-\beta\tau). \end{aligned}$$

Using (6) we also get

$$
\begin{aligned}
UT = BT - BU &= \frac{rp}{a\beta\tau}(1-\beta\tau)\left(\tau+\beta-\frac{a}{p}\tau\right)\\
&= \frac{\Delta}{a\beta\tau}(1-\beta\tau)\left(\frac{p-a}{p}\tau+\beta\right) = \frac{\Delta}{a\beta\tau}(1-\beta\tau)(\beta\gamma\tau+\beta)\\
&= \frac{\Delta}{a\tau}(1-\beta\tau)(1+\gamma\tau).
\end{aligned}
$$

Since $r_1 = UT \cdot \tan\Theta = UT \cdot \tau$, it follows

$$
r_1 = \frac{\Delta}{a}(1-\beta\tau)(1+\gamma\tau). \tag{9}
$$

From (8) and (9) we conclude

$$
BU \cdot UC = ar_1\alpha.
$$

By Theorem 2 it follows that the circle $(P, r_1)$ touches the circle $ABC$ from the inside. The same conclusion holds for the circle $(Q, r_2)$. By the formal substitutions $\beta \leftrightarrow \gamma$ and $\tau \leftrightarrow \frac{1}{\tau}$, from (9) we get the analogous formula

$$
r_2 = \frac{\Delta}{a}\left(1-\frac{\gamma}{\tau}\right)\left(1+\frac{\beta}{\tau}\right) = \frac{\Delta}{a\tau^2}(\tau+\beta)(\tau-\gamma). \tag{10}
$$

□

Since the line $IU$ is normal to the bisector $TP$ of the angle $\angle ATB$ between the lines $AT$ and $BC$, it follows that $U' = IU \cap AT$ is the touching point of the line $AT$ with the circle $(P, r_1)$. This is one of the claims in [9].

By completing the isosceles triangle $\triangle UTU'$ to the rhomb $UTU'X$, it follows that the point $I$ is equally distant from the lines $UT$ and $UX$. Hence, the incircle $(I, r)$ touches the line $UX$, parallel to $AT$ passing through $U$. A similar claim is valid for the parallel to $AT$ passing through $V$. These claims (proved for Thébault's external theorem – see Remark 2) are in the paper [5].

**Remark 1.** Let $I_a$ be the excenter to $BC$ of $\triangle ABC$. By the same construction as in Theorem 3 with $I$ replaced by $I_a$, we get two more circles touching $BC$ and $AT$ and the circle $ABC$ externally.

## 3 Proof of Thébault's theorem

We now give a short proof of Theorem 1 based on our auxiliary results. With the same notations as before we reason as follows:

From (9) and (10), and using (6) we have

$$
\begin{aligned}
r_1 + r_2\tau^2 &= \frac{\Delta}{a}[(1-\beta\tau)(1+\gamma\tau)+(\tau+\beta)(\tau-\gamma)]\\
&= \frac{\Delta}{a}[1-\beta\gamma+\tau^2-\beta\gamma\tau^2]\\
&= \frac{rp}{a}(1-\beta\gamma)(1+\tau^2) = r(1+\tau^2),
\end{aligned}
$$

and this proves formula (2) from Theorem 1. The obtained equality can also be written in the form $r_1 - r = (r - r_2)\tau^2$, or equivalently $\frac{r_1 - r}{r\tau} = \frac{r - r_2}{r}\tau$. By looking at Fig. 3, this is equivalent to

$$\frac{PU - ID}{UD} = \frac{ID - QV}{DV}. \tag{11}$$

This means that the points $P$, $I$ and $Q$ are collinear. Also,

$$\frac{PI}{IQ} = \frac{UD}{DV} = \frac{r\tau}{\frac{r}{\tau}} = \tau^2.$$

## 4 Some related results

In [7] and [8] the collinearity of $P$, $Q$ and $I$ is also proved. In [13], formula (9) is given in the form

$$r_1 = \frac{r}{r_a - r}[r_a - r\tau^2 - (b - c)\tau], \tag{12}$$

and analogously for $r_2$, where $r_a$ is the radius of the excircle to $BC$ of the triangle $\triangle ABC$. Namely, from $r = p\alpha\beta\gamma$, $r_a = p\alpha$, and $a = p(1 - \beta\gamma)$, $b = p(1 - \gamma\alpha)$, $c = p(1 - \alpha\beta)$, it follows $b - c = p\alpha(\beta - \gamma)$. Hence, the right-hand side of (12) is given by

$$\begin{aligned}\frac{r}{\alpha - \alpha\beta\gamma}[\alpha - \alpha\beta\gamma\tau^2 - \alpha(\beta - \gamma)\tau] &= \frac{r}{1 - \beta\gamma}(1 - \beta\gamma\tau^2 - \beta\tau + \gamma\tau)\\ &= \frac{rp}{a}(1 - \beta\tau)(1 + \gamma\tau),\end{aligned}$$

and this is the right-hand side of (9).

**Remark 2.** Let $P'$ and $Q'$ be the centers of the circles touching $BC$ and $AT$ and the circle $ABC$ externally. By the same argument as in the above proof of Thébault's theorem, it follows that $P'$, $I_a$ and $Q'$ are collinear. This was also proved in [5]. This is sometimes called *Thébault's external theorem*.

**Remark 3.** Recall that the general *Appolonius' problem* asks to construct (by ruler and compass) all circles that touch three given circles (possibly of infinite radii) in a plane. Our Theorem 3 and Remark 2 provide a simple solution to a special case of Appolonius' problem when we are given a circle and two of its chords. In fact, many instances of the general Appolonius' problem can be reduced via appropriate inversions to the above case.

**Theorem 4.** ([13]) *With the same notations as in Theorem* 3, *the equality $r_1 = r_2$ holds if and only if the point $T$ coincides with the touching point $D'$ of the line $BC$ and the excircle of the triangle $ABC$ to the side $BC$.*

*Proof.* By using (9) and (10), the equality $r_1 = r_2$ is equivalent to

$$\begin{aligned}&(1 - \beta\tau)(1 + \gamma\tau)\tau^2 = (\tau + \beta)(\tau - \gamma)\\ &\Leftrightarrow \beta\gamma(1 - \tau^4) - \beta\tau(1 + \tau^2) + \gamma\tau(1 + \tau^2) = 0\\ &\Leftrightarrow \beta\gamma(1 - \tau^2) - \beta\tau + \gamma\tau = 0.\end{aligned} \tag{13}$$

From the equalities $BD' = CD = \frac{r}{\gamma}$, $BT = \frac{rp}{a\beta\tau}(\tau + \beta)(1 - \beta\tau)$, and $a = p(1 - \beta\gamma)$, the equality $BT = BD'$ is equivalent to $\gamma(\tau + \beta)(1 - \beta\tau) = \beta\tau(1 - \beta\gamma)$. And as it turns out easily, the last equality is equivalent to (13). □

The circles $(P, r_1)$ and $(Q, r_2)$ touch each other if and only if $UT = TV$. From the proof of Theorem 3, we have $UT = r_1/\tau$, and by substituting $\tau$ by $1/\tau$, it follows $TV = r_2\tau$. So, $UT = TV$ becomes $\tau^2 = r_1/r_2$. Hence, from formula (1) in Theorem 1 we have $PI : IQ = r_1 : r_2$. But this means that the point $I$ is the tangency point of the two circles. Therefore, we have proven the following theorem:

**Theorem 5.** ([6], [11]) *Suppose two circles touch each other externally at the point $I$, they both touch internally the circle $ABC$, both touch at $I$ the line $AI$, and both touch the line $BC$ on the side of the point A. Then $I$ is the incenter of the triangle $\triangle ABC$.*

## 5 Is there any space version of Thébault's theorem?

The main part of Thébault's theorem is the collinearity of the circle centers $P$, $Q$ and $I$, as was claimed in Theorem 1. One would hope the following space version should be true.

**Space version 1.** ("Four spheres with coplanar centers") *Let $I$ be the incenter of a tetrahedron $ABCD$, and let $T$ be any point of the face $\triangle ABC$ (or even the plane $ABC$). Let $P$ be the center of the sphere which touches the three sides of the tetrahedron $TBCD$ (i.e., all except $BCD$) and touches the circumsphere $\Sigma$ of our tetrahedron $ABCD$. The point $Q$ (for the tetrahedron $TACD$), and the point $R$ (for the tetrahedron $TABD$) are defined analogously. Then the four points $P$, $Q$, $R$ and $I$ are coplanar.*

Unfortunately, this is false in general. A counterexample is a 3-sided pyramid $ABCD$, where the base $\triangle ABC$ is a regular triangle of side length $a$ and the altitude is $DT = h$, where $T$ is the center of $\triangle ABC$. The inradius of $ABCD$ is $r = \frac{ah}{a + \sqrt{a^2 + 12h^2}}$, while the radius $\rho$ of the sphere touching the planes $BTC$, $BTD$, $CTD$ and the circumsphere $\Sigma$ of $ABCD$ is given by $\rho = \frac{\sqrt{a^2 + 4h^2} - a}{4h} \cdot a$. It turns out that $\rho \neq r$.

Another "obvious" space version of Thébault's theorem would be the following statement:

**Space version 2.** ("Three spheres with collinear centers") *Let $T$ be a point on the edge $AB$ of a tetrahedron $ABCD$. Let $P$ be the center of the sphere touching the planes $TAC$, $TAD$, $TCD$ and the circumsphere $\Sigma$ of the tetrahedron $ABCD$. Similarly, let $Q$ be the center of the sphere touching the planes $TBD$, $TCD$, $TBC$ and the sphere $\Sigma$. Let $I$ be the incenter of our tetrahedron $ABCD$. Then the points $P$, $I$ and $Q$ are collinear.*

It turns out that this space version is also wrong. A counterexample here is a regular tetrahedron and the midpoint $T$ of one of the edges of the tetrahedron.

So, the question is what is a space version of Thébault's theorem? Is there any reasonable version at all?

## References


[1] Demir, H.; Tezer, C.: Reflections on a problem of V. Thébault. *Geom. Dedicata* 39 (1991), 79–92.

[2] English, B.J.: Solution of Problem 3887. It's a long story. *Amer. Math. Monthly* 110 (2003), 156–158.

[3] Fukagawa, H.: Problem 1260. *Crux Math.* 13 (1987), 181.

[4] Editor's comment. *Crux Math.* 14 (1988), 237–240.

[5] Gueron, S.: Two applications of the generalized Ptolemy theorem. *Amer. Math. Monthly* 109 (2002), 362–370.

[6] Pompe, W.: Solution of Problem 8. *Crux Math.* 21 (1995), 86–87.

[7] Rigby, J.F.: Tritangent circles, Pascal's theorem and Thébault's problem. *J. Geom.* 54 (1995), 134–147.

[8] Roman, N.: Aspura unor problema data la O. I. M. *Gazeta Mat.* (*Bucuresti*) 105 (2000), 99–102.

[9] Seimiya, T.: Solution of Problem 1260. *Crux Math.* 17 (1991), 48.

[10] Shail, R.: A proof of Thébault's theorem. *Amer. Math. Monthly* 108 (2001), 319–325.

[11] Shirali, S.: On the generalized Ptolemy theorem. *Crux Math.* 22 (1996), 49–53.

[12] Stärk, R.: Eine weitere Lösung der Thébault'schen Aufgabe. *Elem. Math.* 44 (1989), 130–133.

[13] Taylor, K.B.: Solution of Problem 3887. *Amer. Math. Monthly* 90 (1983), 487.

[14] Thébault, V.: Problem 3887. Three circles with collinear centers. *Amer. Math. Monthly* 45 (1938), 482–483.

[15] Turnwald, G.: Über eine Vermutung von Thébault. *Elem. Math.* 41 (1986), 11–13.

[16] Veldkamp, G.R.: Een vraagstuk van Thébault uit 1938. *Nieuw Tijdskr. Wiskunde* 61 (1973), 86–89.

[17] Veldkamp, G.R.: Comment. *Crux Math.* 15 (1989), 51–53.



Darko Veljan
Vladimir Volenec
Department of Mathematics
University of Zagreb
Bijenička 30
10 000 Zagreb, Croatia
e-mail: dveljan@math.hr,
volenec@math.hr