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Erdös-Mordell-type inequalities[1]

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The famous Erdös-Mordell inequality states that, if P is a point in the interior of a triangle ABC whose distances are p, q, r from the vertices of the triangle and x, y, z from its sides, then

$$p + q + r \geq 2(x + y + z).$$

In the paper by Satnoianu [1], some generalizations of the above inequality were given. His proof depends heavily on the geometry of the triangle ABC. In this note, we give a more algebraic proof of the Erdös-Mordell inequality.

Theorem. *Let $p, q, r \geq 0$ and let $\alpha + \beta + \gamma = \pi$. Then we have the inequality*

$$p + q + r \geq 2\sqrt{qr}\cos\alpha + 2\sqrt{rp}\cos\beta + 2\sqrt{pq}\cos\gamma\ . \tag{1}$$

Proof. We consider the following quadratic function of x:

$$x^2 - 2(\sqrt{r}\cos\beta + \sqrt{q}\cos\gamma)x + q + r - 2\sqrt{qr}\cos\alpha. \tag{2}$$

Then a quarter of the discriminant is

$$\frac{1}{4}\Delta = (\sqrt{r}\cos\beta + \sqrt{q}\cos\gamma)^2 - (q + r - 2\sqrt{qr}\cos\alpha).$$

Since $\alpha + \beta + \gamma = \pi$, we have

$$\cos\alpha = -\cos(\beta + \gamma) = -\cos\beta\cos\gamma + \sin\beta\sin\gamma.$$

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Using the above identity, the discriminant can be simplified as

$$\Delta = -(\sqrt{r}\sin\beta - \sqrt{q}\sin\gamma)^2 \leq 0.$$

Thus the expression (2) is always nonnegative. Letting $x = \sqrt{p}$, we get (1). □

Corollary. *Let x', y', z' be the length of the angle bisectors of $\angle BPC$, $\angle CPA$, and $\angle APB$, respectively. Then we have*

$$p + q + r \geq 2(x' + y' + z').$$

Proof. We have

$$x' = \frac{2qr}{q+r}\cos\gamma \leq \sqrt{qr}\cos\gamma\,,$$
$$y' = \frac{2pr}{p+r}\cos\beta \leq \sqrt{pr}\cos\beta\,,$$
$$z' = \frac{2pq}{p+q}\cos\alpha \leq \sqrt{pq}\cos\alpha\,.$$

The corollary follows from the theorem. □

Remark. Since $x' \geq x$, $y' \geq y$ and $z' \geq z$, the corollary implies the Erdös-Mordell inequality

$$p + q + r \geq 2(x + y + z).$$

References

[1] Satnoianu, R.: Erdös-Mordell-type inequalities in a triangle. *Amer. Math. Monthly* 110 (2003) 8, 727–729.

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