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**Elemente der Mathematik**

# On summing to arbitrary real numbers

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Following Erdős [1] we say a sequence $\{a_n\}_{n=1}^{\infty}$ is *irrational* if the set $\{\sum_{n\geq 1}\frac{1}{a_n c_n} \mid c_n \in \mathbb{N}\}$, which we refer to henceforth as its *expressible set*, contains no rational numbers. In [1] it is shown that if $\lim_{n\to\infty} a_n^{1/2^n} = \infty$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ then $\sum_{n\geq 1} a_n^{-1}$ is an irrational number. From this Erdős deduces that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is an irrational sequence. Thus its expressible set contains no rational numbers. In [2] it is shown that if $a_n \in \mathbb{R}^+$ for all $n \in \mathbb{N}$ and $\limsup_{n\to\infty} \frac{1}{n}\log_2\log_2 a_n < 1$ then the expressible set of the sequence $\{a_n\}_{n=1}^{\infty}$ contains an interval. It seems to be the case that in general finding the expressible set for the sequence $\{a_n\}_{n=1}^{\infty}$ is not easy.

Ein interessantes zahlentheoretisches Problem ist die Frage nach der Rationalität des Werts einer konvergenten Reihe reeller Zahlen. An diese Fragestellung anknüpfend nennen wir mit P. Erdős eine Folge $\{a_n\}_{n=1}^{\infty}$ reeller Zahlen irrational, falls die Menge $E = \{\sum_{n=1}^{\infty} 1/(a_n c_n) \mid c_n \in \mathbb{N}\}$ keine rationale Zahl enthält. In der vorliegenden Arbeit beweisen die Autoren für den Fall, dass die Reihe $\sum_{n=1}^{\infty} 1/a_n$ bedingt konvergent ist, dass die Menge $E$ jeweils die gesamte reelle Zahlengerade ausschöpft.

In this paper we give conditions on $\{a_n\}_{n=1}^{\infty}$ to ensure that its expressible set is equal to $\mathbb{R}$. We prove the following:

**Theorem 1.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonzero real numbers such that the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is conditionally convergent. Then its expressible set is equal to $\mathbb{R}$.*

A series is conditionally convergent if it is convergent but the series of the absolute values of its terms is not. Theorem 1 is an immediate consequence of the following more general theorem.

**Theorem 2.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonzero real numbers such that the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is conditionally convergent. Then for every pair $\alpha$, $\beta$ of real numbers with $\alpha \leq \beta$ there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that*

$$\alpha = \liminf_{N\to\infty} \sum_{n=1}^{N} \frac{1}{a_n c_n} \qquad \text{and} \qquad \beta = \limsup_{N\to\infty} \sum_{n=1}^{N} \frac{1}{a_n c_n} \,. \tag{1}$$

For the proof of Theorem 2 we need the following two lemmas.

**Lemma 1.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonzero real numbers such that the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is conditionally convergent. Then for every real number $A \geq 0$ and every integer $N \geq 0$ there exist a number $K \in \mathbb{N}$ and numbers $c_{N+1}, \ldots, c_{N+K} \in \mathbb{N}$ such that*

$$\sum_{n=N+1}^{N+K} \frac{1}{a_n c_n} \in \left( A,\ A + \frac{1}{a_{N+K}} \right].$$

*Proof.* Define $\mathcal{P} = \{n \mid a_n > 0\}$ and $\mathcal{N} = \{n \mid a_n < 0\}$. The series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is conditionally convergent, hence

$$\sum_{\substack{n=N+1 \\ n\in\mathcal{P}}}^{\infty} \frac{1}{a_n} = \infty \,.$$

This implies that there exists a positive integer $K$ such that

$$\sum_{\substack{n=N+1 \\ n\in\mathcal{P}}}^{N+K-1} \frac{1}{a_n} \leq A \qquad \text{and} \qquad \sum_{\substack{n=N+1 \\ n\in\mathcal{P}}}^{N+K} \frac{1}{a_n} > A \,.$$

The fact that

$$0 < \sum_{\substack{n=N+1 \\ n\in\mathcal{P}}}^{N+K} \frac{1}{a_n} - \sum_{\substack{n=N+1 \\ n\in\mathcal{P}}}^{N+K-1} \frac{1}{a_n} = \frac{1}{a_{N+K}}$$

immediately gives

$$s = \sum_{\substack{n=N+1 \\ n\in\mathcal{P}}}^{N+K} \frac{1}{a_n} \in \left( A,\ A + \frac{1}{a_{N+K}} \right].$$

Now consider two cases:

(1) Assume that $\mathcal{N} \cap \{N+1, \dots, N+K\} = \emptyset$. In this case put $c_n = 1$ for every $n = N+1, \dots, N+K$ and the result follows.

(2) Now suppose that

$$r = \sum_{\substack{n=N+1 \\ n \in \mathcal{N}}}^{N+K} \frac{1}{a_n} < 0\,.$$

Put $C = \left[\frac{r}{A-s}\right] + 1$. Then

$$0 > \sum_{\substack{n=N+1 \\ n \in \mathcal{N}}}^{N+K} \frac{1}{Ca_n} = \frac{1}{C} \sum_{\substack{n=N+1 \\ n \in \mathcal{N}}}^{N+K} \frac{1}{a_n} > \frac{A-s}{r} \cdot r = A - s\,.$$

Hence the result follows by taking $c_n = 1$ for $n \in \{N+1, \dots, N+K\} \cap \mathcal{P}$ and $c_n = C$ for $n \in \{N+1, \dots, N+K\} \cap \mathcal{N}$. □

**Lemma 2.** *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of nonzero real numbers such that the series* $\sum_{n=1}^{\infty} \frac{1}{a_n}$ *is conditionally convergent. Then for every real number* $A \le 0$ *and every integer* $N \ge 0$ *there exist a number* $K \in \mathbb{N}$ *and numbers* $c_{N+1}, \dots, c_{N+K} \in \mathbb{N}$ *such that*

$$\sum_{n=N+1}^{N+K} \frac{1}{a_n c_n} \in \left[A - \left|\frac{1}{a_{N+K}}\right|,\ A\right).$$

*Proof.* Using the transformation $a_n \mapsto -a_n$ and Lemma 1 we obtain Lemma 2. □

*Proof of Theorem* 2. In the following we set

$$S_k = \sum_{n=1}^{k} \frac{1}{a_n c_n}\,.$$

If $\beta \ge 0$ then putting $A = \beta$ and $N = 0$ into Lemma 1 we obtain a number $K$ and a sequence $\{c_n\}_{n=1}^{K}$ such that

$$S_K \in \left(\beta,\ \beta + \frac{1}{a_K}\right].$$

Then set $N_0 = 0$ and $N_1 = K$.

Similarly, if $\beta < 0$ then $\alpha < 0$, and putting $A = \alpha$ and $N = 0$ into Lemma 2 we get $K$ and $\{c_n\}_{n=1}^{K}$ with

$$S_K \in \left[\alpha - \left|\frac{1}{a_K}\right|,\ \alpha\right).$$

Then set $N_0 = K$.

Now we will construct the sequence $\{c_n\}_{n=1}^{\infty}$ by induction. Consider two cases:

(1) Suppose that we have constructed the sequence $\{N_m\}_{m=0}^{2t+1}$, $t \in \mathbb{N}_0$, with

$$S_{N_{2t+1}} \in \left(\beta,\ \beta + \frac{1}{a_{N_{2t+1}}}\right].$$

Lemma 2 implies that there exist $K$ and $\{c_n\}_{n=N_{2t+1}+1}^{N_{2t+1}+K}$ such that

$$\sum_{n=N_{2t+1}+1}^{N_{2t+1}+K} \frac{1}{a_n c_n} \in \left[\alpha - S_{N_{2t+1}} - \left|\frac{1}{a_{N_{2t+1}+K}}\right|,\ \alpha - S_{N_{2t+1}}\right).$$

Let $N_{2t+2} = N_{2t+1} + K$. Then we have

$$S_{N_{2t+2}} \in \left[\alpha - \left|\frac{1}{a_{N_{2t+2}}}\right|,\ \alpha\right).$$

(2) Suppose that we have constructed the sequence $\{N_m\}_{m=0}^{2t}$, $t \in \mathbb{N}_0$, with

$$S_{N_{2t}} \in \left[\alpha - \left|\frac{1}{a_{N_{2t}}}\right|,\ \alpha\right).$$

Lemma 1 implies that there exist $K$ and $\{c_n\}_{n=N_{2t}+1}^{N_{2t}+K}$ such that

$$\sum_{n=N_{2t}+1}^{N_{2t}+K} \frac{1}{a_n c_n} \in \left(\beta - S_{N_{2t}},\ \beta - S_{N_{2t}} + \frac{1}{a_{N_{2t}+K}}\right].$$

Let $N_{2t+1} = N_{2t} + K$. Then we have

$$S_{N_{2t+1}} \in \left(\beta,\ \beta + \frac{1}{a_{N_{2t+1}}}\right].$$

Using alternatively cases (1) and (2) we construct the whole sequence $\{c_n\}_{n=1}^{\infty}$. From the construction it follows that

- $\alpha - \left|\frac{1}{a_k}\right| \le S_k \le \beta + \left|\frac{1}{a_k}\right|$ for every $k \ge N_1$,
- $S_{N_{2t}} < \alpha$ for every $t \in \mathbb{N}$,
- $S_{N_{2t+1}} > \beta$ for every $t \in \mathbb{N}_0$.

The series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is conditionally convergent, hence $\frac{1}{a_n} \to 0$. This implies that

$$\lim_{t\to\infty} S_{N_{2t}} = \alpha \quad \text{and} \quad \lim_{t\to\infty} S_{N_{2t+1}} = \beta$$

and the result follows. □

## References

[1] Erdős, P.: Some problems and results on the irrationality of the sum of infinite series. *J. Math. Sci.* 10 (1975), 1–7.

[2] Hančl, J.: Expression of Real Numbers With the Help of Infinite Series. *Acta Arith.* LIX.2, (1991), 97–104.

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