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On the Archimedean or semiregular polyhedra

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1 Introduction

1.1 Regular polyhedra

Polyhedra have fascinated mathematicians for at least two and a half millennia. In particular, the *regular* or *platonic* solids were used in Greek astronomy and philosophy in addition to mathematics. Their beauty and symmetries have stimulated investigations that even today are thriving. Our paper deals with a small but fundamental result in their theory.

A *polyhedron* may be intuitively conceived as a "solid figure" bounded by plane faces and straight line edges so arranged that every edge joins exactly *two* (no more, no less) vertices and is a common side of two faces.

A polyhedron is *regular* if all its *faces* are regular polygons (with the same number of sides) and all its *vertices* are regular polyhedral angles; that is to say, all the face angles

Eines der schönsten Ergebnisse der klassischen Raumgeometrie ist die Klassifikation der regulären Polyeder. Es dürfte den meisten Lesern wohlbekannt sein, dass diese Polyeder durch die fünf Platonischen Körper gegeben sind. Ein besonders eleganter Beweis dafür kann mit Hilfe von Eulers Polyederformel gegeben werden. Weniger bekannt ist möglicherweise die Klassifikation der sogenannten halbregulären Polyeder, deren Oberfläche zwar auch aus regelmässigen Vielecken besteht, allerdings können diese nun unterschiedliche Eckenzahlen aufweisen. Dieses Klassifikationsproblem wurde bereits durch Archimedes gelöst: es führt auf die dreizehn halbregulären Polyeder sowie auf die unendlichen Familien von Prismen und Antiprismen. Im nachfolgenden Beitrag gibt der Autor einen elementaren Beweis dieses Resultats unter Verwendung der Eulerschen Polyederformel.

at every vertex are congruent and all the dihedral angles are congruent. An immediate consequence of the definition is that all the faces of the polyhedron are congruent.

There are five such regular convex polyhedra, a fact known since Plato's time, at least, and all of Book XIII of Euclid is devoted to proving it, as well as showing how to construct them: the *tetrahedron*, the *cube*, the *octahedron*, the *dodecahedron*, and the *icosahedron*.

1.2 Archimedean/semiregular polyhedra

It is reasonable to ask what happens if we *forego* some of the conditions for regularity. Archimedes [1] investigated the polyhedra that arise if we *retain* the condition that the faces have to be regular polygons, but *replace* the regularity of the polyhedral angles at each vertex by the *weaker* condition that they all be congruent (see Lines [6]). Such solids are called *Archimedean* or *semiregular* polyhedra.

Theorem 1 (Archimedes' theorem). *There are thirteen semiregular polyhedra as well as two infinite families: the prisms and the antiprisms.*

In the following paper we will prove Archimedes' theorem by elementary topological arguments based on Euler's polyhedral formula (see §2.2). After some simple introductory lemmas the entire proof boils down to solving an inequality involving the number of sides of the polygons that meet at each vertex by an exhaustive enumeration of cases (see §4).

2 Proof techniques

2.1 Euclid's proof for regular polyhedra

Euclid's proof (Proposition XVIII, Book XIII) is based on the *polyhedral angle inequality*: *the sum of the face angles at a vertex cannot exceed* 2π, as well as on the fact that *the internal angle of a regular* p*-gon is* $\pi - \frac{2\pi}{p}$.

Thus, if q faces meet at each vertex

$$\Rightarrow q\left(\pi - \frac{2\pi}{p}\right) < 2\pi \tag{2.1.1}$$

$$\Rightarrow (p-2)(q-2) < 4 \tag{2.1.2}$$

$$\Rightarrow (p,q) = (3,3),\ (4,3),\ (3,4),\ (5,3),\ (3,5) \tag{2.1.3}$$

which give the tetrahedron, cube, octahedron, dodecahedron, and icosahedron respectively.

Of course the key step is to obtain (2.1.2). Euclid does it by (2.1.1) which expresses a *metrical* relation among angle measures.

One presumes that Archimedes applied more complex versions of (2.1.1) and (2.1.2) to prove that the semiregular solids are those thirteen already listed. Unfortunately, his treatise was lost over two thousand years ago!

2.2 Euler's polyhedral formula for regular polyhedra

Almost the same amount of time passed before somebody came up with an entirely new proof of (2.1.2), and therefore of (2.1.3). In 1752 Euler [4] published his famous *polyhedral formula*:

$$\boxed{V - E + F = 2} \tag{2.2.1}$$

in which $V :=$ the number of vertices of the polyhedron, $E :=$ the number of edges, and $F :=$ the number of faces. This formula is valid for any polyhedron that is homeomorphic to a sphere.

The proof of (2.1.2) using (2.2.1) goes as follows. If q p-gons meet at each vertex,

$$\Rightarrow pF = 2E = qV \tag{2.2.2}$$

$$\Rightarrow E = \frac{qV}{2},\ F = \frac{qV}{p}. \tag{2.2.3}$$

Substituting (2.2.3) into (2.2.1),

$$\Rightarrow V - \frac{qV}{2} + \frac{qV}{p} = 2 \quad \Rightarrow \quad 2pV - qpV + 2qV = 4p$$

$$\Rightarrow V = \frac{4p}{2p - qp + 2q} \quad \Rightarrow \quad 2p - qp + 2q > 0 \quad \Rightarrow \quad (p-2)(q-2) < 4$$

which is (2.1.2).

This second proof proves much more. We have found *all regular maps* (graphs, networks) on the surface of a sphere whatever the boundaries may be, without *any* assumptions in regard to they are being circles or skew curves. Moreover the exact shape of the sphere is immaterial for our statements, which hold on a cube or any homeomorph of the sphere.

This *topological* proof of (2.1.2) is famous and can be found in numerous accessible sources, for example Rademacher and Toeplitz [7].

2.3 Proofs of Archimedes' theorem

Euclidean-type *metrical* proofs of Archimedes' theorem are available in the literature (see Cromwell [2] and Lines [6]) and take their origin in a proof due to Kepler [5].

They use the polyhedral angle inequality to prove:

- at most *three* different kinds of face polygons can appear around any solid angle;
- three polygons of different kinds *cannot* form a solid angle if any of them has an *odd* number of sides.

One then exhaustively examines all possible cases.

The situation is quite different with respect to a *topological* proof of Archimedes' theorem. Indeed, after we had developed our own proof, as presented in this paper, we were able to find only one reference: T.R.S. Walsh [8] in 1972.

His proof, too, is based exclusively on Euler's polyhedral formula, and so there are overlaps with ours. However, our proof is quite different, both in arrangement and details, and in purpose. The pedagogical side is insisted upon in our proof so as to make it as elementary and self-contained as possible for as wide an audience as possible. We comment further on the structure of our proof in §4.

3 Three lemmas

For any polyhedron we define:

V:= total number of vertices;

V_p:= total number of vertices incident with p edges;

E:= total number of edges;

F:= total number of faces;

F_p:= total number of p-gonal faces.

Here, and from now on, *polyhedron* means any map on the sphere for which Euler's theorem holds.

3.1 Lemma 1

The following lemma is due to Euler [4] and is well-known. We sketch the proof for completeness.

Lemma 1. *The following relations are valid in any polyhedron:*

1. $3F_3 + 2F_4 + F_5 = 12 + 2V_4 + 4V_5 + \cdots + F_7 + 2F_8 + \cdots$.
2. *At least one face has to be a triangle, or a quadrilateral, or a pentagon; i.e., there is no polyhedron whose faces are all hexagons, or polygons with six or more sides.*

Proof. For 1. we note

(i) $F_3 + F_4 + \cdots + F_{V-1} = F$;

(ii) $3F_3 + 4F_4 + \cdots + (V-1)F_{V-1} = 2E$;

(iii) $V_3 + V_4 + \cdots + V_{F-1} = V$;

(iv) $3V_3 + 4V_4 + \cdots + (F-1)V_{F-1} = 2E$.

Now multiply (i) by 6, subtract (ii), and use (iii), (iv), and Euler's formula.

For 2, observe that F_3, F_4, and F_5 cannot all be zero in 1. at the same time. □

3.2 Definition of semiregular polyhedron. Lemma 2

Definition 1. A polyhedron is called *Archimedean* or *semiregular* if the cyclic order of the degrees of the faces surrounding each vertex is the *same* to within rotation and reflection ([8]).

Lemma 2. *In any Archimedean polyhedron:*

1.
$$\boxed{rV = 2E}$$

where r *edges are incident at each vertex.*

2.
$$\boxed{\frac{pF_p}{q} = V}$$

where q p*-gons are incident at each vertex.*

3.
$$\boxed{V = \frac{2}{1 - \frac{r}{2} + \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_r}}}$$

where the p_k *are the degrees of the* r *polygons meeting at each vertex.*

Proof. For 1., since there are 2 vertices on any edge, the product rV counts each edge twice, so is equal to $2E$.

For 2., pF_p counts the total number of vertices *once* if *one* p-gon is incident at each vertex, *twice* if *two* p-gons are incident there, ..., q times if q p-gons are incident at the vertex. That is, $pF_p = qV$.

For 3., solve 1. for E, use (i) of the proof of Lemma 1.1, solve 2. for F_p, substitute in Euler's formula, solve for V, and write any fraction

$$\frac{q}{p} = \underbrace{\frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p}}_{q-\text{times}}.$$ □

3.3 Lemma 3

This lemma limits the number of candidate polygons surrounding each vertex.

Lemma 3. *If* r *edges are incident with each vertex of an Archimedean polyhedron then*

$$\boxed{r \leqslant 5.}$$

Proof. By 3. of Lemma 2

$$1 - \frac{r}{2} + \frac{1}{p_1} + \cdots + \frac{1}{p_r} > 0 \Rightarrow \frac{1}{p_1} + \cdots + \frac{1}{p_r} > \frac{r-2}{2}.$$

But,

$$p_1 \geqslant 3,\ p_2 \geqslant 3, \dots,\ p_r \geqslant 3 \Rightarrow \frac{1}{3} + \frac{1}{3} + \cdots + \frac{1}{3} \geqslant \frac{1}{p_1} + \cdots + \frac{1}{p_r} > \frac{r-2}{2}$$
$$\Rightarrow \frac{r}{3} > \frac{r-2}{2} \Rightarrow r < 6 \Rightarrow r \leqslant 5. \qquad \square$$

4 The methodology of the topological proof

It is of interest to compare the method of proof, using Euler's theorem, for the *regular* polyhedra and the *Archimedean* polyhedra.

In *both* cases the essential step is to use the fact that *the denominator of the formula for the number of vertices, V, is positive*:

$$V = \frac{2}{1 - \frac{r}{2} + \frac{r}{p}} \qquad \text{regular,}$$
$$V = \frac{2}{1 - \frac{r}{2} + \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_r}} \qquad \text{Archimedean.}$$

In the case of the *regular* polyhedron the inequality

$$1 - \frac{r}{2} + \frac{r}{p} > 0$$

can be rearranged into the elegant inequality

$$(p-2)(r-2) < 4,$$

which, as we saw before, leads to five solutions (p, r).

Unfortunately, in the case of the *Archimedean* polyhedra the inequality

$$1 - \frac{r}{2} + \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_r} > 0$$

apparently does *not* lend itself to an algebraic rearrangement into a product, and so must be studied by *an exhaustive enumeration of cases*.

Nevertheless, it is worth emphasizing that the basic structure of the two arguments is the same at the core, although the elaboration of the cases in the Archimedean case demands some topological counting arguments that are not entirely trivial (see §5.2.1 and §5.3.1).

5 Topological proof of Archimedes' theorem

By Lemma 3 we have to consider three cases:

Case 1: Five faces meet at a vertex: $r = 5$.

Case 2: Four faces meet at a vertex: $r = 4$.

Case 3: Three faces meet at a vertex: $r = 3$.

5.1 Case 1. Five faces meet at a vertex: $r = 5$

By Lemma 3.2,

$$1 - \frac{5}{2} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} = \frac{2}{V} > 0$$

$$\Rightarrow \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} - \frac{3}{2} > 0\ . \qquad (5.1.1)$$

By Lemma 1.2, at least one of $p_1, \ldots, p_5$ has to be 3, 4, or 5.

5.1.1 At least one face is a triangle: $p_1 = 3$

Assuming $p_1 = 3$,

$$\Rightarrow \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} - \frac{3}{2} + \frac{1}{3} > 0 \Rightarrow \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} - \frac{7}{6} > 0\ .$$

Without loss of generality, we assume that:

$$\begin{aligned}
&3 \leqslant p_2 \leqslant p_3 \leqslant p_4 \leqslant p_5 \\
&\Rightarrow \frac{1}{3} \geqslant \frac{1}{p_2} \geqslant \frac{1}{p_3} \geqslant \frac{1}{p_4} \geqslant \frac{1}{p_5} \\
&\Rightarrow \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{p_5} - \frac{7}{6} > 0 \\
&\Rightarrow \frac{1}{p_5} - \frac{1}{6} > 0 \\
&\Rightarrow p_5 < 6 \\
&\Rightarrow p_5 = 5,\ 4,\ 3 \\
&\Rightarrow (p_1, p_2, p_3, p_4, p_5) = (3, p_2, p_3, p_4, 5), (3, p_2, p_3, p_4, 4), (3, p_2, p_3, p_4, 3).
\end{aligned}$$

However, if we take $p_2 \geqslant 3$, $p_3 \geqslant 3$, $p_4 \geqslant 4$, $p_5 \geqslant 4$, then

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} \leqslant \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{3}{2},$$

and this contradicts (5.1.1). Therefore we are left with only three quintuplets:

$$\boxed{(p_1, p_2, p_3, p_4, p_5) = (3, 3, 3, 3, 5), (3, 3, 3, 3, 4), (3, 3, 3, 3, 3).} \qquad (5.1.2)$$

These correspond, respectively, to the *snub dodecahedron*, the *snub cube*, and the *icosahedron*, a regular polyhedron. Using the *C & R symbol* [3] to abbreviate the above quintuplets we are left with:

$$\boxed{\begin{aligned}(p_1, p_2, p_3, p_4, p_5) &= 3^4.5 \quad \text{snub dodecahedron} \\ &= 3^4.4 \quad \text{snub cube} \\ &= 3^5 \quad\ \ \text{regular icosahedron}\end{aligned}} \qquad (5.1.3)$$

5.1.2 All faces have at least four sides: $p_1 \geqslant 4$

It is easy to show that $p_5 < 2$ so that no possibilities exist.

5.2 Case 2. Four faces meet at a vertex: $r = 4$

By Lemma 2.3,

$$1 - \frac{4}{2} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} > 0 \Rightarrow \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} - 1 > 0.$$

Again, at least one of the p_k must be 3, 4, or 5.

5.2.1 At least one face is a triangle: $p_1 = 3$

We will write p, q, r instead of p_2, p_3, p_4. Thus the inequality becomes

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{2}{3} > 0. \tag{5.2.1}$$

We examine a typical polyhedron:

- it must have a triangle at each vertex;
- there must be 4 edges incident at each vertex;
- the vertices must all have the same configuration in the same order to within rotation and reflection.

Consider Fig. 1. As we label the faces around each vertex of the triangle ABC, say counterclockwise, from the vertex A, we see that the sequence $(3, p, q, r)$ at A, or its reflection $(3, r, q, p)$, must repeat itself, in that order at B, and then at C. But CB is then an edge of a polygon with r sides and with p sides simultaneously, i.e., we conclude that $p = r$. This means that we are compelled to conclude that *no matter how we label the vertices, at least two of the p, q, r must be equal.*

Here instead of using sides or angles to classify the polyhedral faces, one uses the number of vertices or edges to classify the polygons.

Putting $r = p$ in the inequality (5.2.1), we obtain

$$\begin{aligned} &\frac{2}{p} + \frac{1}{q} - \frac{2}{3} > 0 \\ &\Rightarrow (p-3)(2q-3) < 9 \\ &\Rightarrow 1 < 2q - 3 < 9,\ (2q-3)\ \text{odd} \\ &\Rightarrow 2q - 3 = 3,\ 5,\ 7. \end{aligned}$$

If $2q - 3 = 5$ or 7 then $p - 3 = 0, 1$ resp. $p = 3, 4$. Otherwise, if $2q - 3 = 3$ then $p - 3 = 0, 1, 2$ resp. $p = 3, 4, 5$.

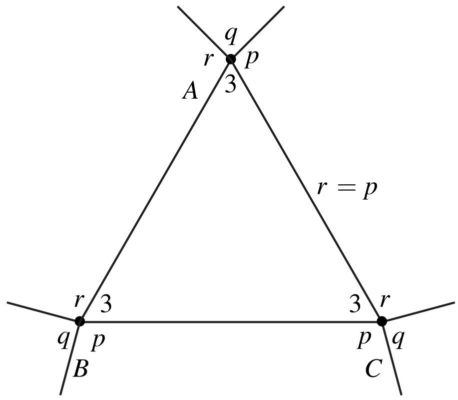

Fig. 1 Vertex constraint

Therefore, we obtain

p	3	3	3	4	4	4	5
q	3	4	5	3	4	5	3

Finally we observe that $2q-3 \geqslant 9$ is permitted if $p-3=0$.
Therefore, we are left with:

$(p,q)=(4,5) \Rightarrow (p_1,p_2,p_3,p_4)=(3.4.5.4)$		small rombicosidodecahedron
$(p,q)=(5,3) \Rightarrow (p_1,p_2,p_3,p_4)=(3.5)^2$		icosidodecahedron
$(p,q)=(4,4) \Rightarrow (p_1,p_2,p_3,p_4)=3.4^3$		small rhombicuboctahedron
$(p,q)=(4,3) \Rightarrow (p_1,p_2,p_3,p_4)=(3.4)^2$		cuboctahedron
$(p,q)=(3,3) \Rightarrow (p_1,p_2,p_3,p_4)=3^4$		regular octahedron
$(p,q)=(3,m) \Rightarrow (p_1,p_2,p_3,p_4)=3^3.m\ (m \geqslant 4)$		antiprism

5.2.2 All faces have at least four sides: $p_1 \geqslant 4$

If we assume that $4 \leqslant p_1 \leqslant p_2 \leqslant p_3 \leqslant p_4$, then

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}-1 \leqslant 0.$$

Therefore $p_1 \geqslant 4$ cannot happen.
There are no other cases with $r=4$.

5.3 Case 3. Three faces meet at a vertex: $r = 3$

By Lemma 2.3,

$$1-\tfrac{3}{2}+\tfrac{1}{p_1}+\tfrac{1}{p_2}+\tfrac{1}{p_3}>0 \Rightarrow \tfrac{1}{p_1}+\tfrac{1}{p_2}+\tfrac{1}{p_3}-\tfrac{1}{2}>0\ .$$

Since at least one of the p_k must be equal to 3, 4, or 5, we consider each case separately.

5.3.1 At least one face is a triangle: $p_1 = 3$

Then,

$$\frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{6} > 0.$$

Looking at the configuration we see:

- each vertex has three edges incident to it,
- two are the edges of a triangle and the third of a p_3-gonal face.

Labeling it we see that

$$p_2 = p_3,$$

and therefore the above equality becomes

$$\frac{2}{p_3} - \frac{1}{6} > 0 \Rightarrow p_3 < 12, \quad 3 \leqslant p_3 \leqslant 11.$$

Lemma 4. *p_3 is even or $p_3 = 3$.*

Proof. We look at the configuration with $p_3 \geqslant 4$. Since the vertices must all look alike, as we traverse counterclockwise (say) the p_3 vertices of a p_3-gonal face, we observe that the edges of the face fall into two groups:

- those that are the common edge of two p_3-gonal faces;
- those that are the common edge of a triangle and a p_3-gonal face.

Moreover, they occur in adjacent pairs, and finally, as we complete one circuit and return to our starting point, having started with a triangular edge, we end up with an edge common to two p_3-gonal faces. Thus we traverse an *integral number of pairs of sides* as we run through the p_3-gonal face once, i.e., p_3 is *even*. □

The only even numbers p_3 between 3 and 11 are

$$p_3 = 4, \; 6, \; 8, \; 10.$$

Therefore we obtain

$p_3 = 3$	$\Rightarrow$	$(p_1, p_2, p_3) = 3^3$	regular tetrahedron
$p_3 = 4$	$\Rightarrow$	$(p_1, p_2, p_3) = 3.4^2$	triangular prism
$p_3 = 6$	$\Rightarrow$	$(p_1, p_2, p_3) = 3.6^2$	truncated tetrahedron
$p_3 = 8$	$\Rightarrow$	$(p_1, p_2, p_3) = 3.8^2$	truncated cube
$p_3 = 10$	$\Rightarrow$	$(p_1, p_2, p_3) = 3.10^2$	truncated dodecahedron

5.3.2 All faces have at least four sides and one exactly four sides: $p_1 = 4 \leqslant p_2 \leqslant p_3$

Then,

$$\frac{1}{4} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{2} > 0 \Rightarrow (p_2 - 4)(p_3 - 4) < 16.$$

The same sort of configuration argument shows *that p_2 and p_3 are even*, and we conclude

$(p_1, p_2, p_3) = (4.6.10)$	great rhombicosidodecahedron
$(p_1, p_2, p_3) = (4.6.8)$	great rhombicuboctahedron
$(p_1, p_2, p_3) = 4.6^2$	truncated octahedron
$(p_1, p_2, p_3) = 4^3$	cube
$(p_1, p_2, p_3) = 4^2.m\ (m \geqslant 4)$	prism

We note that this subcase covers precisely the polyhedra with *bipartite* graphs. Here the vertex set V is the union of two disjoint sets V_1 and V_2, and each edge of the graph goes from V_1 to V_2. Equivalently, each p_k is *even*.[1]

5.3.3 All faces have at least five sides and one exactly five sides: $p_1 = 5 \leqslant p_2 \leqslant p_3$

This is quite similar the the previous section. Since

$$5 = p_1 \leqslant p_2 \leqslant p_3 \Rightarrow \frac{1}{p_2} + \frac{1}{p_3} - \frac{3}{10} > 0 \Rightarrow (3p_2 - 10)(3p_3 - 10) < 100.$$

Again, a configuration argument shows that

$$p_2 = p_3 \Rightarrow (3p_2 - 2)^2 < 100 \Rightarrow 15 \leqslant 3p_2 < 20 \Rightarrow p_2 = 5,\ 6,$$

which gives

$(p_1, p_2, p_3) = 5^3$	regular dodecahedron
$(p_1, p_2, p_3) = 5.6^2$	truncated icosahedron

And we have completed the topological proof of Archimedes' theorem.

We have *not* demonstrated that the polyhedra enumerated in Archimedes' theorem are in fact *constructible*. Again, this is done in the works of Cromwell [2] and Lines [6].

6 Final remarks

As in the case of the topological proof that there are five regular polyhedra, we have proven much more! We have found all *semiregular maps* on any homeomorph of the sphere, a result of great generality. Although the metric proofs are of great interest, intrinsically and historically, the topological proof shows that they appeal to unessential properties of their metric realizations and that, at the root of it all, Archimedes' theorem is a consequence of certain combinatorial relations among the numbers of vertices, edges, and faces.

One wonders what Archimedes would have thought of our proof of his theorem. We hope that he would have liked it.

[1] We thank Michael Josephy for this observation.

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