Elem. Math. 63 (2008) 88 – 96
0013-6018/08/020088-9

Elemente der Mathematik

On maximum area polygons in a planar point set

Yatao Du and Ren Ding

Yatao Du received her M.Sc. and Ph.D. degrees from Hebei Normal University in China. At present, she is teaching at Shijiazhuang Mechanical Engineering College in China. Her research interests focus primarily on discrete and combinatorial geometry.

Ren Ding is a professor of mathematics and head of a Ph.D. program at Hebei Normal University in China. His research interests focus primarily on discrete and combinatorial geometry.

Let P be a finite planar point set with no three points collinear, i.e. in general position. A subset $Q \subset P$ is called a *convex polygon in* P if Q forms the vertex set of a convex polygon. A convex polygon $Q \subset P$ is called an *empty convex polygon* if there is no point of P in the interior of the convex hull of Q. Denote the area of the convex hull of $Q \subset P$ by $S(Q)$. Let

$$f_k(P) =: \max\left\{\frac{S(Q)}{S(P)} : Q \text{ is an empty convex } k\text{-gon with vertices in } P\right\},$$
$$f_k(n) =: \min\{f_k(P) : |P| = n,\ P \text{ is in general position}\}.$$

A finite set of points in the plane is called in *convex position* if it forms the set of vertices of a convex polygon. Let P be a finite set of points in convex position in the plane. Then a polygon Q with vertices in P is always an empty polygon. Let

$$f_k^{\text{conv}}(n) =: \min\{f_k(P) : |P| = n,\ P \text{ is in convex position}\}.$$

Es sei P die Eckenmenge eines konvexen n-Ecks in der Ebene, und es sei $S(P)$ dessen Flächeninhalt. Wird eine k-elementige Teilmenge $Q \subset P$ dieser Ecken ausgewählt, so überdeckt das zugehörige k-Eck den Bruchteil $S(Q)/S(P)$ der Gesamtfläche. Man wird versuchen, durch geeignete Wahl von Q diesen Flächenanteil m öglichst gross zu machen. In dem nachfolgenden Beitrag behandeln die Autoren das folgende Minimax-Problem: Welcher Bruchteil $S(Q)/S(P)$ l ässt sich, unabhängig von der Form des Ausgangspolygons, durch geeignete Wahl von Q garantiert erreichen? Beispielsweise finden die Autoren im Fall $|P| = 5$ und $|Q| = 4$, dass das Viereck bei richtiger Wahl der weggelassenen Ecke mindestens den Bruchteil $2/(5-\sqrt{5})$ des Fünfecks überdeckt. Eine allgemeine Lösung des hier behandelten Problems wäre wünschbar.

In [2] the authors studied $f_3^{\text{conv}}(n)$. In this paper we evaluate $f_4^{\text{conv}}(n)$, and more generally, $f_{n-1}^{\text{conv}}(n)$.

Theorem 1. $f_4^{\text{conv}}(5) = \frac{2}{5-\sqrt{5}}$.

Proof. Let P be a convex 5-gon with vertices A, B, C, D, E in clockwise order. Suppose that the 4-gon $ABCD$ is a maximum area quadrilateral in P. Given two triangles, there exists a unique affine transformation which transforms one triangle into another. So, without loss of generality, we may assume that $A = (0,0)$, $B = (0,1)$, $D = (1,0)$, $C = (a,b)$ $(a > 0, b > 0)$. We always assume that $b \geq 1$, see Fig. 2. Indeed, when $b < 1$, the distance from B to the straight line AD is greater than the distance from C to the straight line AD, then we can reflect P about a vertical line without changing the ratio of the areas. See Fig. 1. Relabel the vertices of P to ensure that the distance from C' to the straight line $A'D'$ is greater than distance from B' to the straight line $A'D'$, and in this way we come to the case of $b \geq 1$.

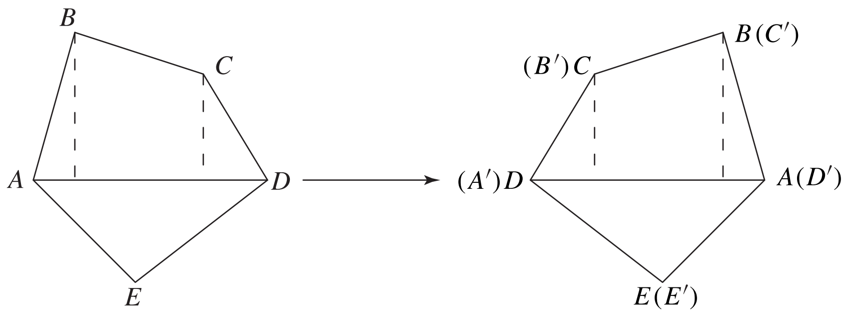


Fig. 1 The assumption $b \geq 1$

Let Q_1, Q_2 and Q_3 denote the 4-gons $ABCD$, $ABDE$ and $ACDE$, respectively, see Fig. 2. Let f be the line through A and C, and f' be the parallel line through D. Similarly, let g be the line through B and D, and g' be the parallel line through A. Since $Q_1 = ABCD$ is a maximum area quadrilateral in P, so E lies completely above f' and g'. Let $F = f' \cap g'$, then $F = (\frac{b}{a+b}, \frac{-b}{a+b})$ and $E \in \triangle ADF$, and hence P is always contained in the convex 5-gon $P' = ABCDF$. Since $b \geq 1$, we have $S(Q_3) \geq S(Q_2)$; and since $S(Q_1) \geq S(Q_3)$, we have $S(\triangle ABC) \geq S(\triangle ADE)$. Suppose $E = (x_0, y_0)$,

$$S(\triangle ABC) = \frac{a}{2},\ S(\triangle ADE) = \frac{-y_0}{2} \Longrightarrow \frac{a}{2} \geq \frac{-y_0}{2} \Longrightarrow y_0 \geq -a.$$

So E lies above the horizontal line $h : y = -a$. See Fig. 2, where E does not appear since its position is depending. The figure shows only the case where F lies below the line h.

Case 1. Suppose F lies above the line h, then $\frac{-b}{a+b} \geq -a$, that is $\frac{b}{a+b} \leq a$. Notice that $P \subset P'$ and so $S(P) \leq S(P')$.

$$S(Q_1) = \frac{1}{2}(a+b),\quad S(P') = \frac{1}{2}(a+b) + \frac{b}{2(a+b)}. \qquad (*)$$

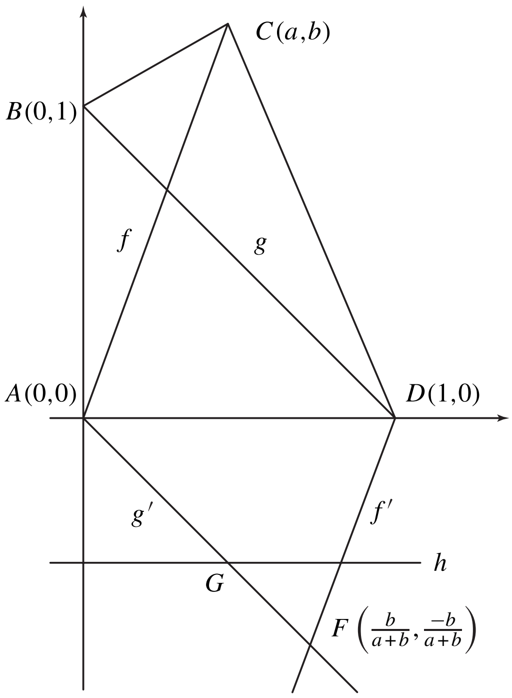


Fig. 2 $E \in \triangle ADF$, $Q_1 = ABCD$, $Q_2 = ABDE$, $Q_3 = ACDE$, $P' = ABCDF$, $P'' = ABCDG$, $h : y = -a$

Subcase 1.1. Suppose $\frac{b}{a+b} \le \frac{a}{b}$, then $\frac{a}{b} \ge \frac{\sqrt{5}-1}{2}$. By $(*)$ and $b \ge 1$ we have

$$\frac{S(P)}{S(Q_1)} \le \frac{S(P')}{S(Q_1)} = 1 + \frac{b}{(a+b)^2} \le 1 + \frac{b^2}{(a+b)^2} = 1 + \frac{1}{(\frac{a}{b}+1)^2} \le \frac{5-\sqrt{5}}{2}$$

$$\Longrightarrow \frac{S(Q_1)}{S(P)} \ge \frac{2}{5-\sqrt{5}} .$$

Subcase 1.2. Suppose $\frac{b}{a+b} > \frac{a}{b}$, then

$$\frac{b}{a} > \frac{2}{\sqrt{5}-1} \Longrightarrow \frac{a}{a+b} < \frac{3-\sqrt{5}}{2}.$$

Recall that $\frac{b}{a+b} \le a$, we have

$$\frac{S(P)}{S(Q_1)} \le \frac{S(P')}{S(Q_1)} \le 1 + \frac{b}{(a+b)^2} \le 1 + \frac{a}{a+b} \le \frac{5-\sqrt{5}}{2}$$

$$\Longrightarrow \frac{S(Q_1)}{S(P)} \ge \frac{2}{5-\sqrt{5}} .$$

Case 2. Suppose F lies below the line h (see Fig. 2), then $\frac{-b}{a+b} < -a$, that is $\frac{b}{a+b} > a$, and since E is above the horizontal line h, so $S(P) \leq S(P'')$, where $P'' = ABCDG$ is a 5-gon with $G = g' \cap h$. Since $g' : \ y = -x, h : \ y = -a$, so $G = (a, -a)$.

$$S(P'') = \frac{1}{2}(a+b) + \frac{1}{2}a.$$

$$\frac{b}{a+b} > a \Longrightarrow \frac{b}{a+b} > \frac{a}{b} \Longrightarrow \frac{b}{a} > \frac{2}{\sqrt{5}-1} \Longrightarrow \frac{a}{a+b} < \frac{3-\sqrt{5}}{2}.$$

$$\frac{S(P)}{S(Q_1)} \leq \frac{S(P'')}{S(Q_1)} = 1 + \frac{a}{a+b} \leq \frac{5-\sqrt{5}}{2} \Longrightarrow \frac{S(Q_1)}{S(P)} \geq \frac{2}{5-\sqrt{5}} .$$

From the above argument, we obtain that for any 5-point set P in convex position we have $f_4(P) \geq \frac{2}{5-\sqrt{5}}$ and hence $f_4^{\text{conv}}(5) \geq \frac{2}{5-\sqrt{5}}$.

Let $a = \frac{\sqrt{5}-1}{2}$, $b = 1$, and hence the line h passes through F. Let $E = F$, then $\frac{S(Q_1)}{S(P)} = \frac{a+b}{a+b+\frac{b}{a+b}} = \frac{2}{5-\sqrt{5}}$ by $(*)$, so $f_4^{\text{conv}}(5) \leq \frac{2}{5-\sqrt{5}}$.

Hence $f_4^{\text{conv}}(5) = \frac{2}{5-\sqrt{5}}$. □

Theorem 2. $f_4^{\text{conv}}(6) \geq \frac{1}{4-\sqrt{5}}$.

Proof. Let P be a convex 6-gon with vertices A_1, A_2, A_3, A_4, A_5, A_6 in clockwise order. Suppose that the 4-gon Q is a maximum area quadrilateral in P, then Q must be in one of the forms of $A_iA_{i+1}A_{i+2}A_{i+4}$, $A_iA_{i+1}A_{i+3}A_{i+4}$ or $A_iA_{i+1}A_{i+2}A_{i+3}$, where the addition in the subscript is modulo 6.

Case 1. Suppose $Q = A_iA_{i+1}A_{i+2}A_{i+4}$.

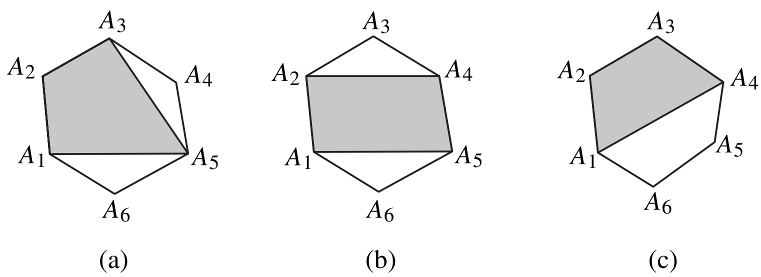


Fig. 3 Possible forms of maximum area quadrilaterals in P

Without loss of generality let $Q = A_1A_2A_3A_5$, as shown in Fig. 3(a). Let $P_1 = A_1A_2A_3A_4A_5$, $P_2 = A_1A_2A_3A_5A_6$. Then Q is also the maximum area quadrilateral in P_1 and in P_2. By Theorem 1, we have

$$\frac{S(P)}{S(Q)} = \frac{S(P_1)+S(P_2)-S(Q)}{S(Q)} \leq \frac{5-\sqrt{5}}{2} + \frac{5-\sqrt{5}}{2} - 1 = 4-\sqrt{5}.$$

Thus

$$\frac{S(Q)}{S(P)} \geq \frac{1}{4-\sqrt{5}}.$$

Case 2. Suppose $Q = A_i A_{i+1} A_{i+3} A_{i+4}$, see Fig. 3($b$). By the same argument as in Case 1 we obtain the required conclusion.

Case 3. Suppose $Q = A_i A_{i+1} A_{i+2} A_{i+3}$, that is, Q is formed by four consecutive vertices of P. Without loss of generality let $Q = A_1 A_2 A_3 A_4$, as shown in Fig. 3(c). By using an affine transformation we may assume that $A_1 = (0, 0)$, $A_2 = (0, 1)$, $A_4 = (1, 0)$, $A_3 = (a, b)$ $(a > 0, b \geq 1)$. See Fig. 4.

Let f be the line through A_1 and A_3, and let f' be the parallel line through A_4. Similarly, let g be the line through A_2 and A_4, and let g' be the parallel line through A_1. Let $F = f' \cap g'$, so $F = (\frac{b}{a+b}, \frac{-b}{a+b})$. Similar to the proof of Theorem 1, here $A_5, A_6 \in \triangle A_1 A_4 F$ and A_5, A_6 lie above the horizontal line $h : y = -a$. So P must be contained in the convex 5-gon $P' = A_1 A_2 A_3 A_4 F$.

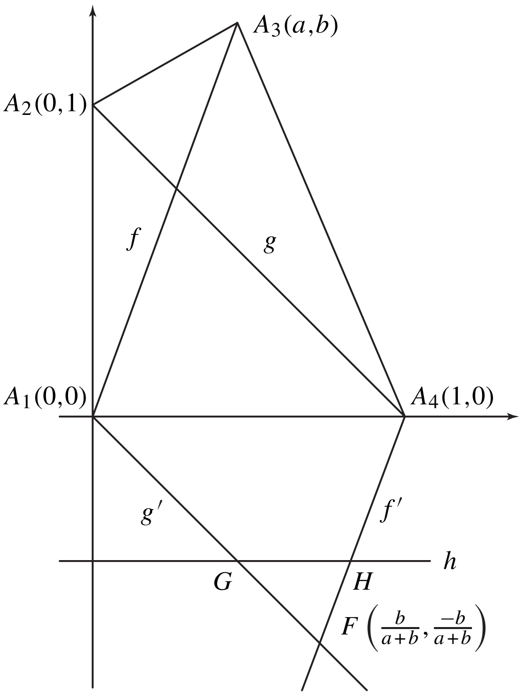


Fig. 4

Subcase 3.1 Suppose F lies above the line h, then $\frac{-b}{a+b} \geq -a$, i.e. $\frac{b}{a+b} \leq a$, so A_5 and A_6 obviously lie above the line h. By the same argument as in Case 1 in proving Theorem 1, we have

$$\frac{S(P)}{S(Q)} \leq \frac{5-\sqrt{5}}{2} < 4-\sqrt{5} \Longrightarrow \frac{S(Q)}{S(P)} > \frac{1}{4-\sqrt{5}}.$$

Subcase 3.2 Suppose F lies below the line h, then $\frac{-b}{a+b} < -a$, i.e. $\frac{b}{a+b} > a$, so P must be contained in the hexagon $P'' = A_1A_2A_3A_4HG$ with $G = g' \cap h$ and $H = f' \cap h$, where $G = (a, -a)$, and $H = (1 - \frac{a^2}{b}, -a)$. The area of P'' equals the area of the 4-gon Q plus the area of the 4-gon A_1A_4HG, hence

$$S(P'') = \frac{1}{2}(a+b) + a - \frac{a^3 + a^2 b}{2b},$$

$$\frac{S(P)}{S(Q)} \leq \frac{S(P'')}{S(Q)} = 1 + \frac{2a}{a+b} - \frac{a^3 + a^2 b}{(a+b)b} < 1 + \frac{2a}{a+b} < 4 - \sqrt{5}$$

$$\Longrightarrow \frac{S(Q)}{S(P)} > \frac{1}{4-\sqrt{5}},$$

and hence $f_4^{\text{conv}}(6) \geq \frac{1}{4-\sqrt{5}}$. □

Lemma A. *Let P_n be the set of vertices of a regular n-gon, and let $r_4(n) =: f_4(P_n)$, then*

$$r_4(n) = \frac{4}{n \sin \frac{2\pi}{n}} \quad \textit{when } n \equiv 0 \pmod 4;$$

$$r_4(n) = \frac{3\cos\frac{\pi}{2n} + \cos\frac{3\pi}{2n}}{n \sin \frac{2\pi}{n}} \quad \textit{when } n \equiv 1 \textit{ or } 3 \pmod 4;$$

$$r_4(n) = \frac{4\cos\frac{\pi}{n}}{n \sin \frac{2\pi}{n}} \quad \textit{when } n \equiv 2 \pmod 4.$$

Proof. Suppose that the maximum area quadrilateral $ABCD$ with vertices in P_n divides

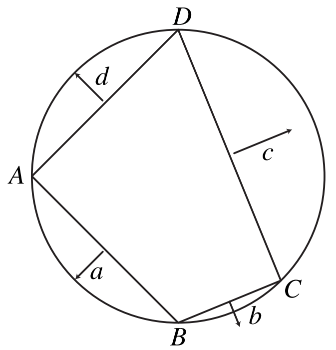


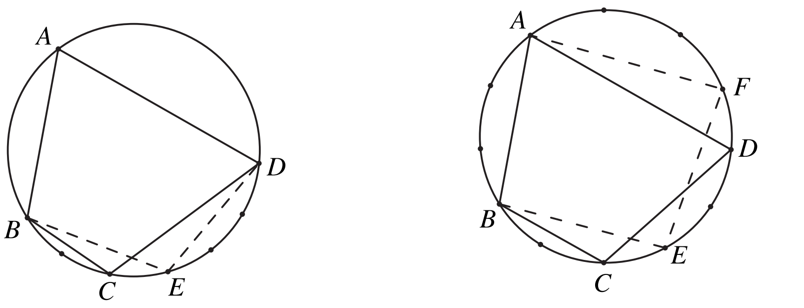
Fig. 5 4 chains on the boundary of convex hull of P_n

the boundary of the convex hull of P_n into four chains $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$ with a, b, c and d edges, respectively, as shown in Fig. 5.

First, we prove that any two of these numbers differ at most by 1.

Case 1. Suppose for two adjacent numbers, say, b and c we have $c-b \geq 2$. See Fig. 6(a).

Let E be the nearest point of P_n to C in anticlockwise order. Observe that since $c-b \geq 2$, the numbers of points of P_n on $\widehat{ED}$ is greater than that on $\widehat{BC}$. $S(\triangle DCE) > S(\triangle BCE)$, because both triangles have the same base CE, and the distance from D to the straight line CE is greater than the distance from B to the straight line CE. Then the area of the 4-gon $ABDE$ is greater than the area of the 4-gon $ABCD$, contradicting the choice of the 4-gon $ABCD$.



(a) $b = 2, c = 4$ (b) $a = c = 3, b = 2, d = 4$

Fig. 6

Case 2. Suppose for two nonadjacent numbers, say, b and d we have $d - b \geq 2$. From Case 1, we need only to consider the cases $a = c = b + 1$, and $d = b + 2$, as shown in Fig. 6(b).

Let E be the nearest point of P_n to C and F be the nearest point of P_n to D in anticlockwise order. Then $S(ABCD) = S(ABED) < S(ABEF)$, contradicting the choice of the 4-gon $ABCD$.

Therefore, we conclude that a maximal area quadrilateral in P_n splits the boundary of the convex hull of P_n into four chains whose numbers of edges are $\{t, t, t, t\}$, $\{t, t, t, t+1\}$, $\{t, t, t+1, t+1\}$, $\{t, t+1, t+1, t+1\}$, when $n \equiv 0, 1, 2, 3 \pmod 4$, respectively. An easy computation leads to the claimed formulas. □

Notice that each $r_4(n)$ is a decreasing function. Thus we can deduce that

$$\lim_{n\to\infty} r_4(n) = \frac{2}{\pi} \ .$$

Lemma B. *Let B be a compact convex region in the plane and B_k be a largest area k-gon inscribed in B. Then $area(B_k) \geq area(B)\frac{k}{2\pi}\sin\frac{2\pi}{k}$, where equality holds if and only if B is an ellipse.*

From Theorem 2, Lemma A and Lemma B, the following results can be easily obtained:

Theorem 3. *For planar point sets in convex position of size $n \geq 7$, we have*

$$\frac{2}{\pi} \leq f_4^{\text{conv}}(n) \leq r_4(n).$$

Theorem 4. $\frac{1}{4-\sqrt{5}} \leq f_4^{\text{conv}}(6) \leq r_4(6) = \frac{2}{3}$.

Theorem 5. $f_{n-1}^{\text{conv}}(n) \leq 1 - \frac{2(1-\cos\frac{2\pi}{n})}{n} \quad (n \geq 4)$.

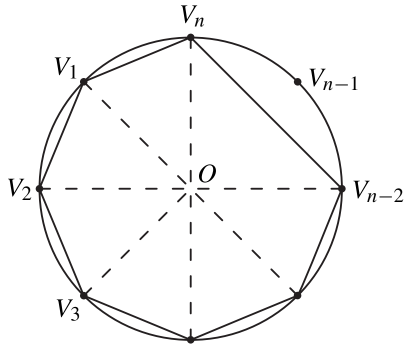


Fig. 7

Proof. Let $P_n = V_1V_2V_3\ldots V_n$ be a regular n-gon with circumradius equal to 1 and circumcenter at O, then every $(n-1)$-gon Q in P_n has the same area. See Fig. 7.

$$S(P_n) = nS(\triangle V_1OV_2) = \frac{n}{2}\sin\frac{2\pi}{n},$$

$$S(Q) = (n-2)S(\triangle V_1OV_2) + S(\triangle V_{n-2}OV_n) = \frac{n-2}{2}\sin\frac{2\pi}{n} + \frac{1}{2}\sin\frac{4\pi}{n}$$

$$\Longrightarrow \frac{S(Q)}{S(P_n)} = 1 - \frac{2(1-\cos\frac{2\pi}{n})}{n}.$$

Hence $f_{n-1}^{\text{conv}}(n) \leq 1 - \frac{2(1-\cos\frac{2\pi}{n})}{n}$ by the definition of $f_{n-1}^{\text{conv}}(n)$. □

A few words on why the topic should be discussed might be necessary. The study of exact algorithms for robot motion planning forms a major subarea of computational geometry, with connections also to symbolic and algebraic computation. Motion planning is useful not only for computer control of actual robots but also for assembly planning and to computer animation. A related problem is found in robot motion planning where one might want to approximate the shape of a robot moving from one room to the next through a narrow door, the numerical bounds for the ratio of the maximum area quadrilateral to the robot body area could be a measure of how good an approximation is. For details see [1].

Acknowledgement. We are very grateful to Prof. Masatsugu Urabe for his great help in sending us his manuscript and some related papers. We are also very grateful to the anonymous referee for his comments and suggestions.

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Yatao Du
Department of Mathematics
Shijiazhuang Mechanical Engineering College
Shijiazhuang, 050003, China
e-mail: dyt77@eyou.com

Ren Ding*
Department of Mathematics
Hebei Normal University
Shijiazhuang, 050016, China
e-mail: rending@hebtu.edu.cn
rending@heinfo.net

*This research was supported by National Natural Science Foundation of China (10571042), NSF A2005000144, 05CXJJ02 and L2004202.