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A simple solution to Ulam's liar game with one lie

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Introduction

We consider the following game between a questioner and a responder, first proposed by Ulam [9]. (A variation of this game was independently proposed by Rényi, see [5].) Both Rényi and Ulam were motivated by questions arising from communication over noisy channels. The responder thinks of an integer $x \in \{1, \dots, n\}$ and the questioner must determine x by asking questions whose answer is 'Yes' or 'No'. The responder is allowed to lie at most k times during the game. Let $q_k(n)$ be the maximum number of questions needed by the questioner, under an optimal strategy, to determine x under these rules. In particular, Ulam asked for the value of $q_1(10^6)$ (as this is related to the well-known 'twenty questions' game). It follows from an observation of Berlekamp [1] that $q_1(10^6) \geq 25$ and

Es wird ein Spiel mit den Spielern A und B betrachtet: A überlegt sich eine Zahl x zwischen 1 und 10^6. B will nun die Zahl x durch Fragen ermitteln, die von A mit „ja“ oder „nein“ beantwortet werden. Es ist leicht zu sehen, dass B mittels binärer Suche die Zahl x mit 20 Fragen herausfinden kann, 19 Fragen aber nicht immer ausreichen. 1976 fragte Ulam, was die von B benötigte Anzahl Fragen ist, wenn A einmal eine falsche Antwort geben darf. Pelc zeigte 1990, dass B den Wert von x mit 25 Fragen ermitteln kann. Berlekamp hatte zuvor beobachtet, dass 24 Fragen nicht immer ausreichen. Die Autoren geben in diesem Artikel einen kurzen Beweis für das Ergebnis von Pelc und betrachten auch etwas allgemeinere Fragestellungen. Die Motivation für die Untersuchung dieses Spiels kommt aus der Kodierungstheorie.

Rivest et al. [6] as well as Spencer [7] gave bounds which imply that $q_1(10^6) \leq 26$. Pelc [4] was then able to determine $q_1(n)$ exactly for all n:

Theorem 1 ([5]) *For even $n \in \mathbb{N}$, $q_1(n)$ is the smallest integer q which satisfies $n \leq 2^q/(q+1)$. For odd $n \in \mathbb{N}$, $q_1(n)$ is the smallest integer q which satisfies $n \leq (2^q - q + 1)/(q+1)$.*

In particular, his result shows that the lower bound of Berlekamp for $n = 10^6$ was correct. Shortly afterwards, Spencer [7] determined $q_k(n)$ asymptotically (i.e. for fixed k and large n). The values of $q_k(10^6)$ have been determined for all k. These and many other related results are surveyed by Hill [3], Pelc [5] and Cicalese [2]. Here, we give a simple strategy and analysis for the game with at most one lie which implies the above result of Pelc for many values of n.

Theorem 2 *If $n \leq 2^\ell \leq 2^q/(q+1)$ for some integer ℓ, then the questioner has a strategy which identifies x in q questions if at most one lie is allowed. In particular, $q_1(n) \leq q$.*

Below, we will give a self contained argument (Proposition 3) which shows that if n also satisfies $n > 2^{q-1}/q$, then the strategy in Theorem 2 is optimal. This implies that the bound in Theorem 2 is optimal if $n = 2^\ell$ for some $\ell \in \mathbb{N}$. More generally, Theorem 1 implies that for even n, Theorem 2 gives the correct bound if and only if we can find a binary power 2^ℓ with $n \leq 2^\ell \leq 2^q/(q+1)$, where q is the smallest integer with $n \leq 2^q/(q+1)$. (Similarly, one can read off a more complicated condition for odd n as well.) In particular, if $n = 10^6$, we obtain $q_1(10^6) = 25$. To check this, note that for $q = 25$ and $\ell = 20$, we have

$$\lceil 2^{q-1}/q \rceil = 671088 < n \leq 1048576 = 2^\ell < 1290555 = \lfloor 2^q/(q+1) \rfloor.$$

If one compares the bounds from Theorems 1 and 2, then one can check that the smallest value where the latter gives a worse bound is $n = 17$, where Theorem 2 requires 9 questions whereas $q_1(17) = 8$. The smaller values are $q_1(2) = 3$, $q_1(3) = q_1(4) = 5$, $q_1(5) = \ldots = q_1(8) = 6$ and $q_1(9) = \ldots = q_1(16) = 7$.

More generally, it is easy to see that for any n the strategy in Theorem 2 uses at most two questions more than an optimal strategy. Indeed, given n, let ℓ and q be the smallest integers satisfying $n \leq 2^\ell \leq 2^q/(q+1)$. So Theorem 2 implies that q questions suffice. Proposition 3 implies that if $n > 2^{q-3}/(q-2)$, then any successful strategy needs at least $q-2$ questions in the worst case. To see that $n > 2^{q-3}/(q-2)$, suppose that this is not the case. Then by assumption on ℓ we have $2^{\ell-1} < n \leq 2^{q-3}/(q-2)$. So if $q \geq 4$ (which we may assume in view of the above discussion of small values), we have $2^\ell < 2^{q-2}/(q-2) \leq 2^{q-1}/q$. This contradicts the choice of q.

Our proof of Theorem 2 uses ideas from Cicalese [2] and Spencer [8]. It gives a flavour of some techniques which are typical for the area. Elsholtz (personal communication) has obtained another short proof for the case $n = 10^6$. Throughout, all logarithms are binary.

From now on, we consider only the game in which at most one lie is allowed. For the purposes of the analysis, it is convenient to allow the responder to play an adversarial strategy, i.e. the responder does not have to think of the integer x in advance (but does answer the questions so that there always is at least one integer x which fits all but at most one of the previous answers). The questioner has then determined x as soon as there is exactly one integer which fits all but at most one of the previous answers. We analyze the game by associating a sequence of states (a, b) with the game. The state is updated after each answer. The variable a denotes the number of integers which fit all previous answers, and b is the number of integers which fit all but exactly one answer. So initially, $a = n$ and $b = 0$. The questioner has won as soon as $a + b \leq 1$. If there are j questions remaining in the game and the state is (a, b), then we associate a weight $w_j(a, b) := (j + 1)a + b$ with this state. Also, we call the integers which fit all answers but exactly one *pennies* (note that each of these contributes exactly one to the weight of the state).

For completeness, we now give a proof of the lower bound mentioned in the introduction. As mentioned above, the fact is due to Berlekamp [1], see also [2, 4, 6] for the argument. The proof has a very elegant probabilistic formulation which generalizes more easily to the case of $k \geq 1$ lies (see Spencer [8]).

Proposition 3 *If $n > 2^{q-1}/q$, then the questioner does not have a strategy which determines x with $q - 1$ questions.*

Proof. Note that our assumption implies that the initial weight satisfies $w_{q-1}(n, 0) > 2^{q-1}$. It is easy to check that before each answer, the sum of the weights of the two possible new states $(a_{\text{yes}}, b_{\text{yes}})$ and $(a_{\text{no}}, b_{\text{no}})$ is equal to the weight of the current state (a, b), i.e.

$$w_j(a, b) = w_{j-1}(a_{\text{yes}}, b_{\text{yes}}) + w_{j-1}(a_{\text{no}}, b_{\text{no}}). \tag{1}$$

To see this, observe that $a = a_{\text{yes}} + a_{\text{no}}$ and $a + b = b_{\text{yes}} + b_{\text{no}}$ and substitute this into the definition of the weight functions. (1) implies that the responder can always ensure that the new state (a', b') (with j questions remaining) satisfies

$$w_j(a', b') \geq w_{j+1}(a, b)/2 \geq w_{q-1}(n, 0)2^{-(q-1-j)} > 2^j. \tag{2}$$

Thus the responder can ensure that the final state has weight greater than one. We also claim that this game never goes into state $(1, 0)$. (Together, this implies that the final state consists of more than one penny, which means that the responder wins). To prove the claim, suppose that we are in state $(1, 0)$ with $j-1$ questions remaining. Then the previous state must have been $(1, t)$ for some $t > 0$. Note that (2) implies that $w_j(1, t) > 2^j$. On the other hand, the assumption on the strategy of the responder implies that $w_{j-1}(1, 0) \geq w_{j-1}(0, t)$. Combined with (1), this means that $w_j(1, t) = w_{j-1}(1, 0) + w_{j-1}(0, t) \leq 2w_{j-1}(1, 0) = 2j$. But $2j < 2^j$ has no solution for $j \geq 1$, and so we have a contradiction. □

Proof of Theorem 2

Note that the weight of the initial state is $w_q(n, 0) = n(q + 1) \leq 2^q$. By making n larger if necessary, we may assume that $\log n = \ell$, for some $\ell \in \mathbb{N}$. So $\ell \leq q - \log(q + 1)$. Since

$\ell \in \mathbb{N}$, this implies

$$\ell \leq q - \lceil \log(q+1) \rceil. \tag{3}$$

Consider each integer $\leq 2^\ell$ in its binary form, i.e. we have 2^ℓ strings of length ℓ. The questioner performs a binary search on these numbers by asking questions of the form 'Is the value of x in position i a 1?'. The binary search on the search space $\{1, \dots, n\}$ uses exactly ℓ questions and as a result we obtain $\ell + 1$ possible numbers for x: There is exactly one integer which satisfies all the answers, and there are also ℓ integers which satisfy all but one answer. Therefore, after the binary search has been performed we are in state $(1, \ell)$. Moreover, $w_{q-\ell}(1, \ell) = 1 \cdot (q - \ell + 1) + \ell \cdot 1 = q + 1$.

Let $p = q - \ell$. By (3), it now suffices to identify x within $p := \lceil \log(q+1) \rceil$ questions. Note that the weight of the state satisfies $2^{p-1} < w_{q-\ell}(1, \ell) \leq 2^p$. Suppose that $q + 1$ is not a power of 2. It is easy to see that we can add pennies to the state until the total weight is equal to 2^p, as the addition of pennies will only make the game harder for the questioner. Suppose that we now have r pennies in total, so we obtain the new state $P^* = (1, r)$, with $r \geq \ell$, where the weight of P^* equals 2^p. Thus

$$p + 1 + r = w_p(1, r) = 2^p. \tag{4}$$

We now have two cases to consider:

Case 1: If $r < p + 1$, then (4) implies that $p + 1 > 2^{p-1}$, which holds if and only if $p \leq 2$. This means that we have one nonpenny and at most two pennies. It is easy to see that the questioner can easily identify x using two more questions in this case.

Case 2: Suppose $r \geq p+1$. This implies that $2^{p-1} \geq p+1$ and thus $p > 2$. We know that the total weight of this state is even and so we wish to find a set, say A_p, such that when a question is asked about it, regardless of the responder's reply, the weight is exactly halved. Assume that A_p contains the nonpenny and y pennies and that the weight of A_p is equal to 2^{p-1}. Suppose that the answer to 'Is $x \in A_p$?' is 'Yes'. Then the weight of the resulting state is $p + y$ (since we are left with one nonpenny of weight p and y pennies). If the answer is 'No', the resulting state has weight $r + 1 - y$ (since the nonpenny has turned into a penny and the y pennies have been excluded). Thus we wish to solve $r + 1 - y = p + y$, which gives

$$y = \frac{1}{2}(r + 1 - p). \tag{5}$$

Note also that (4) implies $r + 1 - p$ is even and so y is an integer. Moreover, the condition $r \geq p + 1$ implies that $y \geq 1$.

So suppose that the questioner chooses A_p as above and asks 'Is $x \in A_p$?'. If the responder replies 'Yes', we obtain a position P', which consists of one nonpenny and y pennies, i.e. $P' = (1, y)$, which has weight 2^{p-1}. If $p - 1 = 2$, then by Case 1, the questioner can easily identify x. If $p - 1 > 2$, we redefine r such that $r := y$ and then calculate the new value of y by (5), to obtain a new set A_{p-1}. The questioner continues inductively with A_{p-1} instead of A_p, so the next question will be 'Is $x \in A_{p-1}$?'. If the responder replies

'No' to the original question 'Is $x \in A_p$?' then we obtain a position P' which consists only of pennies, i.e. $P' = (0, r - y + 1)$. Again, this has weight 2^{p-1}. Since we have $p - 1$ questions remaining we perform a binary search on the $r - y - 1 = 2^{p-1}$ pennies remaining and after $p - 1$ questions we will have identified x.

Note that eventually, the answer to the question 'Is $x \in A_i$?' must be either 'No' or it is 'Yes' and we have $i - 1 = 2$ as well as a new weight of 2^{i-1} (in which case there are 2 questions and at most one nonpenny and two pennies remaining). By the above arguments, the questioner can find the integer x in the required total number q of questions in both cases, which completes the proof of the theorem. □

In case $n = 10^6$, the above strategy would mean that after 20 questions, we would be in state $(1, 20)$ and have weight $w_5(1, 20) = 26$. Our aim is to find x within 5 more questions. We add 6 pennies to obtain the state $(1, r)$ with $r = 26$ and weight 2^p, where $p = 5$. Thus (5) gives $y = 11$. So A_5 consists of the nonpenny and 11 pennies. If the answer is 'Yes', then A_4 consists of the nonpenny and 4 pennies. If the answer is 'No', we have 16 pennies left and can find x after 4 more questions by using binary search.

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