Elem. Math. 63 (2008) 122 – 125
0013-6018/08/030122-4

Elemente der Mathematik

Mean value theorems for differences

J. Marshall Ash

J. Marshall Ash received his Ph.D. from the University of Chicago in 1966. Since 1974 he is a full professor at DePaul University in Chicago. He has been a visiting professor at Stanford University. His research interests comprise generalized differentiation, trigonometric polynomials, and uniqueness of representation by multiple trigonometric series.

1 One dimension

The mean value theorem says that if $f(x)$ has a derivative at every point $x \in (a, b)$ and is continuous at $x = a$ and $x = b$, then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We investigate the truth of a finite difference variant of this result, namely whether, given f continuous on $[a, b]$ and given $p \in (0, b - a)$, there is a $c \in [a, b - p]$ such that

$$\frac{f(c + p) - f(c)}{p} = \frac{f(b) - f(a)}{b - a}.$$

Aus dem Analysisunterricht ist uns der Mittelwertsatz der Differentialrechnung wohlbekannt. In dem nachfolgenden Beitrag beweist der Autor ein Analogon für Differenzenquotienten. Dazu sei f eine stetige reellwertige Funktion auf dem abgeschlossenen Intervall $[a, b]$. Weiter werde vorausgesetzt, dass $p \in (0, b - a)$ ein echter Teiler von $b - a$ ist, d.h. dass es eine natürliche Zahl $n > 1$ mit $b - a = n \cdot p$ gibt. Dann garantiert der Mittelwertsatz für Differenzenquotienten die Existenz eines $c \in [a, b - p]$, so dass die Gleichung

$$\frac{f(c + p) - f(c)}{p} = \frac{f(b) - f(a)}{b - a}$$

besteht. Darüber hinaus gibt der Autor Gegenbeispiele zu diesem Satz, wenn p kein Teiler von $b - a$ ist.

First, suppose p is a proper divisor of $b-a$, i.e., $p = \frac{b-a}{n}$ for some integer $n \geq 2$. Let $m = \frac{f(b)-f(a)}{b-a}$ and $r(x) = \frac{f(x+p)-f(x)}{p}$. Then

$$\frac{1}{n}\big(r(a) + r(a+p) + \ldots + r(a+(n-1)p)\big) = \frac{f(b)-f(a)}{np} = m. \tag{1.1}$$

All $r(a+ip) > m$ would force the left hand side to be $> m$, so some $r(a+ip) \leq m$; similarly some $r(a+jp) \geq m$. The intermediate value theorem assures us that the continuous function r must take on the value m for some x between $a+ip$ and $a+jp$.

Second, suppose that p is not a proper divisor of $b-a$ so that $b-a = np+\epsilon$ for some natural number n and some $\epsilon \in (0,p)$. Set $g(x) = \sin^2 \pi \frac{x-a}{p} - \frac{x}{b-a}\sin^2 \pi \frac{b-a}{p}$. Then $\frac{g(b)-g(a)}{b-a} = 0$; but for every $c \in [a, b-p]$, $\frac{g(c+p)-g(c)}{p} = -\frac{1}{b-a}\sin^2 \pi \frac{b-a}{p} < 0$. We have shown:

Theorem 1. *Let f be a real-valued continuous function on a closed interval $[a,b]$ and let $m = \frac{f(b)-f(a)}{b-a}$. If $p = (b-a)/n$ for some integer $n \geq 2$, then there is a $c \in [a, b-p]$ so that $\frac{f(c+p)-f(c)}{p} = m$. However, if $p \in (0, b-a)\setminus\left\{\frac{b-a}{2}, \frac{b-a}{3}, \ldots\right\}$, then there is an infinitely differentiable function $g = g_p$ so that for every $c \in [a, b-p]$, $\frac{g(c+p)-g(c)}{p} \neq \frac{g(b)-g(a)}{b-a}$.*

Remark 1. The negative side of this result manifests itself in a couple of counterintuitive facts. One is that it is possible for runner A to run a marathon at a perfectly steady 8 minute per mile pace and for runner B to run that marathon so that *every* mile interval $[x, x+1]$, $0 \leq x \leq 25.2$ is run in 8 minutes and 1 second but so that B beats A ([5, Problem 167]). The other is that it is possible for a runner to run 1609 meters at an average rate of speed that exceeds his average rate of speed for *every* interval of the form $[x, x+1600]$, $0 \leq x \leq 9$ ([1]). Comparing these two phenomena motivated this paper. We now know that the connection between these facts is that 1 is not a proper divisor of 26.2 and 1600 is not a proper divisor of 1609.

The point c in the statement of the mean value theorem is strictly interior to $[a,b]$. If $b-a = np$ with the integer $n \geq 3$ we can similarly find c so that $[c, c+p]$ is strictly interior to $[a,b]$. For the proof given above produces such a c except when $r(a) = m$; while if $r(a) = m$, either all $r(a+ip) = m$ whence $c = a+p$ works, or $[r(a+ip)-m][r(a+jp)-m] < 0$ for some $i, j \geq 1$ whence a satisfactory c strictly between $a+ip$ and $a+jp$ can be found. However the $n=2$ case is different: for example, if $n=2$, $[a,b] = [0, 2\pi]$, and $f(x) = \sin x$; then $[c, c+p] = [c, c+\pi]$ cannot be chosen to be strictly interior to $[0, 2\pi]$.

When the original interval $[a,b]$ is replaced by a circle's circumference, the conclusion becomes very different. Identify $[a,b)$ with the circumference of a circle and say that f is *almost continuous* if f is continuous at each point of $[a,b)$ and if $f(b^-) = \lim_{h\searrow 0} f(b-h)$ exists. An arc $\widehat{\alpha\beta}$ of length p, $p < b-a$ corresponds either to an interval of the form $[\alpha, \beta]$ if $a \leq \alpha < \beta \leq b$ where $\beta = \alpha + p$ or to the union of $[\alpha, b]$ and $[a, \beta]$ when $(b-\alpha) + (\beta - a) = p$.

Theorem 2. *Let f be almost continuous on the circle $[a,b)$. Then for every $p \in (0, b-a)$ there is an arc $\widehat{\alpha\beta}$ of length p so that*

$$\frac{f(b^-) - f(a)}{b-a} = \frac{f(\beta) - f(\alpha)}{p},$$

where $f(\beta)$ must be taken to be $f(b^-)$ when $\alpha + p = b$.

Let $m = \frac{f(b^-)-f(a)}{b-a}$ and $f^*(x) = f(x) - mx$ for $x \in [a,b)$. Extend f^* to $\mathbb{R}$ by making it $(b-a)$-periodic. Then f^* is continuous at b and hence continuous. Integration over a period is independent of the starting point, so

$$\int_a^b \left\{ f^*(x+p) - f^*(x) \right\} dx = 0.$$

Since f^* is continuous, the integrand must be 0 at some point x_0. So if $\widehat{\alpha\beta}$ is the arc determined by x_0 and $x_0 + p$, $0 = f^*(x_0 + p) - f^*(x_0) = f(\beta) - f(\alpha) - mp$.

2 Higher dimensions

There is also a d-dimensional analogue of all this. Everything works inductively and easily, so we restrict our discussion to $d = 2$. Fix a function $f : \mathbb{R}^2 \to \mathbb{R}$. By a *box* we mean a closed non-degenerate rectangle with sides parallel to the axes. For a box $B := [a, a+P] \times [b, b+Q]$, an analogue of the mean value theorem asserts that if f is continuous on B and if f_{xy} exists on the interior of B, then there is a point (r,s) interior to B so that

$$\frac{\Delta B}{PQ} = f_{xy}(r,s)$$

where $\Delta B = f(a+P, b+Q) + f(a,b) - f(a+P, b) - f(a, b+Q)$. The proof of this is a straightforward induction ([2, Proposition 2; also 4]). (In d dimensions, ΔB becomes an alternating sum of the evaluations of f at the 2^d vertices of a d-dimensional cuboid, PQ becomes the volume of that cuboid, and f_{xy} becomes $f_{x_1 x_2 \ldots x_d}$.) The analogue of our original question becomes this.

Question. *Let $(p,q) \in (0,P) \times (0,Q)$ be given. Must there be a box $b \subset B$ of dimensions $p \times q$ so that*

$$\frac{\Delta B}{PQ} = \frac{\Delta b}{pq}\ ?$$

The answer is just what you would expect: "yes" if (p,q) is in $\left\{\frac{P}{2}, \frac{P}{3}, \frac{P}{4}, \ldots\right\} \times \left\{\frac{Q}{2}, \frac{Q}{3}, \frac{Q}{4}, \ldots\right\}$, and "no" otherwise. To prove the "yes" part first notice that if B is a finite union of nonoverlapping boxes B_i, then $\Delta B = \sum_i \Delta B_i$; then proceed as in the one dimensional proof by writing $\frac{\Delta B}{PQ} = \frac{\Delta B}{|B|}$ as an average of $\frac{PQ}{pq}$ terms $\frac{\Delta B_i}{|B_i|}$. A counterexample when $P = np + \epsilon$ for some natural number n and some $\epsilon \in (0,p)$

is $y\left(\sin^2\pi\frac{x-a}{p} - \frac{x}{P}\sin^2\pi\frac{P}{p}\right)$, and there is a similar counterexample when Q is not a proper multiple of q.

Some history: If Theorem 1 is called “the Mean Value Theorem for Differences”, then the corresponding result when $f(a) = f(b) = 0$ might be called “Rolle’s Theorem for Differences”. As in the infinitesimal case, the two results are quite equivalent. The positive part of the theorem above appeared in 1806 and the negative part, at least for Rolle’s Theorem for Differences, in 1934. See [3], where Rolle’s Theorem for Differences is called “the Universal Chord Theorem”, for these facts and many more. I thank R. Narasimhan for calling my attention to the very entertaining reference [3].

References

[1] Ash, G.; Ash, J.M.; and Catoiu, S.: Linearizing mile run times. *College Math. J.* 35 (2004), 370–374.

[2] Ash, J.M.; Cohen, J.; Freiling, C.; and Rinne, D.: Generalizations of the Wave Equation. *Trans. Amer. Math. Soc.* 338 (1993), 57–75.

[3] Boas, R.P.: A Primer of Real Functions. *Carus Math. Monogr.* No. 13, Washington, D.C. 1981.

[4] Bögel, K.: Über die mehrdimensonale Differentiation. *Jahresber. Deutsche Math.-Verein.* 65 (1962), 45–71.

[5] Konhauser, J.D.E.; Velleman, D.; and Wagon, S.: Which way did the bicycle go. *Dolciani Math. Exp.* No. 18, Washington, D.C. 1996.

J. Marshall Ash*
Mathematics Department
DePaul University
Chicago, IL 60614, USA
e-mail: mash@math.depaul.edu

*This research was supported by a grant from the Faculty and Development Program of the College of Liberal Arts and Sciences, DePaul University.