
A dynamic proof of Thébault's theorem

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Both authors have studied in Innsbruck and are experts in the numerical solution of differential equations. The present article is a fruit of the joint elaboration of the book project *Geometry by Its History*.

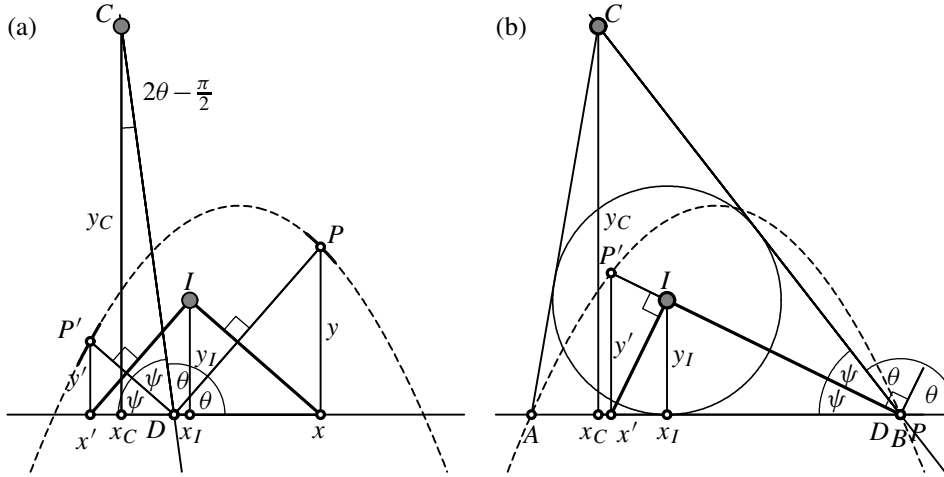
“This computer proof took some 44 hours of CPU on a symbolic 3600 machine (. . .). The theorem almost has the status of a benchmark problem in Groebner theory.” (R. Shail [3])

Thébault's theorem, discovered in 1938 (see [5]), has the remarkable property that it required three decades to obtain a first proof, which then took 24 pages of calculations. Subsequent shorter proofs, but not easy ones, were published mainly in *Elemente der Mathematik* ([4] and [6]) and in Dutch language. We refer to [1], [2], and [7] for complete accounts of all these and recent proofs. In this note we want to emphasize the role of a certain parabola which comes out of a dynamic machine, and which allows an easy understanding.

1 The TTT-machine¹

We start with a “machine” defined as follows (see Fig. (a)): *Choose two fixed points C and I in the plane and let the line CD rotate around C , where D moves on the x -axis. Let our*

Für den schönen Satz von Thébault über den In- und Umkreis eines Dreiecks, welcher in den Dreissigerjahren des letzten Jahrhunderts als Problem gestellt wurde und drei Jahrzehnte ohne Beweis blieb, gibt es inzwischen etliche Beweise, welche in der Regel auf analytischen Rechnungen beruhen, die sich über einige Seiten erstrecken. Zwei etwas kürzere Beweise und weitere Untersuchungen zu diesem Gegenstand führten seit 1986 zu vier Publikationen in den *Elementen der Mathematik*. Die Motivation der Autoren zu dem vorliegenden, weiteren Beweis des Thébaultschen Theorems besteht darin, die Aussage dieses Satzes „auf einen Blick“ zu erkennen. Dieses Ziel erreichen sie durch Anwendung von Sätzen des Apollonius und der Verwendung einer Parabel, die durch eine sogenannte „TTT-Maschine“ (die drei T's stehen für Thébault, Turnwald und Tinguely) erzeugt wird.



machine then produce the two angle bisectors at D (which are mutually orthogonal) and project the point I orthogonally to these bisectors onto the x -axis, producing the abscissas x and x' , respectively. Then the corresponding points P and P' on the bisectors move on one and the same parabola with vertical axis.

Proof. We denote the coordinates of I by (x_I, y_I) , and those of C by (x_C, y_C) . Taking the slope of DP , $p = \tan \theta$, as parameter, we have, because of $\tan 2\theta = \frac{2p}{1-p^2}$, the expression

$$x_D = x_C + y_C \frac{p^2 - 1}{2p} = x_C + \frac{y_C}{2} \left(p - \frac{1}{p} \right)$$

for the abscissa of D . We further have by construction

$$\begin{aligned} \text{(I)} \quad x &= x_I + p y_I, & y &= p(x - x_D), \\ \text{(II)} \quad x' &= x_I - \frac{1}{p} y_I, & y' &= -\frac{1}{p}(x' - x_D). \end{aligned}$$

We next insert the formula for x_D into (I) to obtain

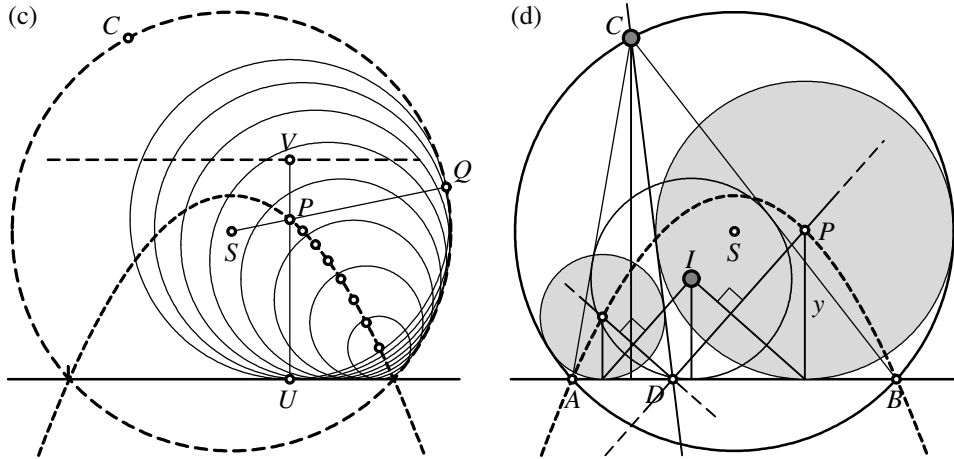
$$y = px - px_D = px - px_C - \frac{y_C}{2}(p^2 - 1).$$

Elimination of p with $p = \frac{x-x_I}{y_I}$ then leads to a quadratic expression for y in x . Interchanging p and $-\frac{1}{p}$ leaves x_D invariant and turns equations (I) into equations (II). Therefore, we obtain precisely the same quadratic expression for y' and x' . \square

¹The first T stands for Thébault, the second for Turnwald, whose corrected version of a formula of Thébault was the main motivation for this machine, the third T stands for Jean Tinguely and emphasizes the dynamic thinking of our proof.

2 The incircle

If $y_C > 2y_I$, our parabola is curved downwards and intersects the x -axis in two points A and B . These points A and B are characterized by the property that I is the incenter of the triangle ABC (see Fig. (b)). We see this, if we let the point D move towards B , say. The point P' , and hence also the point I , then lie on the angle bisector of the angle ABC (and similarly for the other side).



3 The enveloping circle

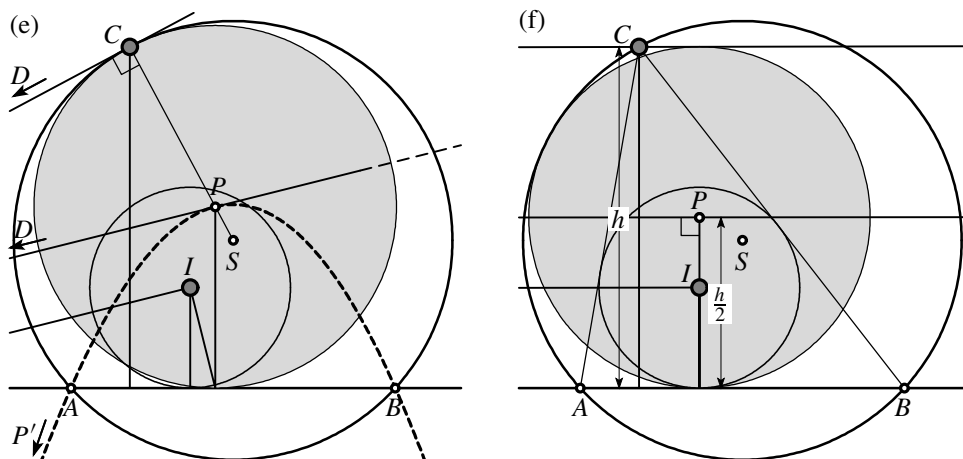
We now consider the family of circles centered at points P on the parabola and tangent to the x -axis, i.e., of radius $PQ = PU = y$ (see Fig. (c)). The enveloping curve of these circles is a circle centered in S , the focus of the parabola. This is the reciprocal result of the fact that the points, which have the same distance from a straight line and a circle, lie on a parabola. Indeed, since by the characterization of the parabola, $SP = PV$, we have that $SQ = UV = \text{Const}$. So each of these circles touches the enveloping circle in the corresponding point Q , which is positioned in the prolongation of SP .

4 The circumcircle

As, by construction, the points P and P' lie on the angle bisectors of CD with the x -axis, and, at the same time, have the same distances from the x -axis and the enveloping circle of Fig. (c), the circles centered at P and P' with radius y are

- tangent to the x -axis;
- tangent to CD ;
- and tangent to the enveloping circle of Fig. (c)

(see Fig. (d)). We now continue to turn our machine until the tangent CD , which rotates around C , becomes orthogonal to CS (see Fig. (e)). In this case, the circle centered in P can only touch the line CD in the point $Q = C$ and the enveloping circle must therefore pass through C . Since it also passes through A and B (here all distances are equal to 0), we conclude that this circle is identical with the circumcircle of the triangle ABC .



5 Thébault's theorem: The points P , I , and P' are aligned.

Proof. By taking appropriate linear combinations of (I) and (II) we obtain

$$\frac{1}{p}x + px' = \left(\frac{1}{p} + p\right)x_I, \quad \frac{1}{p}y + py' = x - x' = \left(\frac{1}{p} + p\right)y_I$$

from which the result follows at once. □

More precisely, the proof shows that the point I intersects² the segment PP' in the ratio $p : \frac{1}{p}$. We can also say that I is the center of gravity for the masses $\frac{1}{p}$ and p attached to P and P' , respectively.

6 The $h/2$ -circle

We add two nice particular cases of the above theorems, firstly: *If the incenter of a triangle is moved upwards to half of the altitude, and the radius of the incircle is increased accordingly, then the resulting circle is tangent to the circumcircle.*

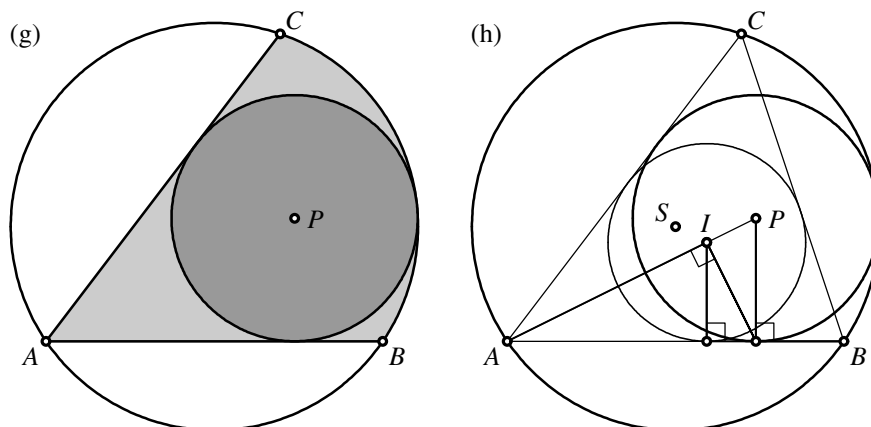
Proof. We see this result by letting $p \rightarrow 0$; in this case D tends to $-\infty$ and the angle bisector tends to the horizontal line of altitude $y_C/2$ (see Fig. (f)). The projection $I \mapsto P$ becomes vertical. □

7 The biggest chocolate egg in a bag

We answer the following question: If an angle BAC at the periphery of a circle cuts from this circle a triangular shaped region (see Fig. (g)), we ask for the largest circle which fits into this set.

² This observation has first been made by G. Turnwald [6] in the form $y \cos^2 \theta + y' \sin^2 \theta = y_I$, correcting a wrong assertion of V. Thébault. This formula was the starting motivation for our “machine”.

Solution: We obtain the answer by turning our machine such that D coincides with A . One has to project the incenter I of the triangle ABC orthogonally to AI onto the side AB , and then orthogonally to AB back to the angle bisector, which gives the center P of the required circle (see Fig. (h)).



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