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Elemente der Mathematik

Algebraic numbers of the form $P(T)^{Q(T)}$ with T transcendental

Diego Marques

Diego Marques was awarded a Ph.D. at the Universidade de Brasília in 2009. Recently he obtained a professorship in the department of mathematics at the same university. His field of interest is transcendental number theory.

1 Introduction

When I was a high-school student, I liked writing rational numbers as "combination" of irrational ones, for instance

$$2 = \sqrt{\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}} = \sqrt{3}^{\frac{\log 4}{\log 3}} = e^{\log 2}.$$

In particular, the last equality above shows us one way of writing the algebraic number 2 as power of two transcendental numbers. In 1934, the mathematicians A.O. Gelfond [2] and T. Schneider [3] proved the following well-known result: If $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and $\beta \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, then α^β is a transcendental number. This result, named as Gelfond-Schneider theorem, classifies completely the arithmetic nature of the numbers of the form $A_1^{A_2}$, for $A_1, A_2 \in \overline{\mathbb{Q}}$. Returning to our subject, but now using the Gelfond-Schneider theorem, we also can easily write 2 as T^T, for some T transcendental. Actually, all prime numbers and

Im Jahr 1934 lösten A.O. Gelfond und T. Schneider das siebte Hilbertsche Problem, indem sie zeigten, dass für algebraische Zahlen α, β mit $\alpha \neq 0, 1$ und $\beta \notin \mathbb{Q}$ die Grösse α^β, also z.B. $\sqrt{2}^{\sqrt{2}}$, transzendent ist. Eine Art Umkehrung dieses Sachverhalts bedeutet die Fragestellung, unter welchen Bedingungen an zwei transzendente Zahlen σ, τ die Grösse σ^τ algebraisch ist. Beispielsweise sind die Eulersche Zahl $e = 2,71828\ldots$ und $\log(2)$ transzendent, aber es ist $e^{\log(2)} = 2$. In der vorliegenden Arbeit zeigt der Autor, dass es zu zwei beliebigen, nicht-konstanten Polynomen $P(X)$ und $Q(X)$ mit rationalen Koeffizienten jeweils unendlich viele algebraische Zahlen gibt, die in der Form $P(\tau)^{Q(\tau)}$ mit transzendentem τ dargestellt werden können.

all algebraic numbers $A \geq e^{-1/e}$, satisfying $A^n \notin \mathbb{Q}$ for all $n \geq 1$, can be written in this form; for a more general result see [4, Proposition 1]. Using again the Gelfond-Schneider theorem and Galois theory, we show that for all non-constant polynomials $P(x)$, $Q(x) \in \mathbb{Q}[x]$, there are infinitely many algebraic numbers which can be written in the particular "complicated" form $P(T)^{Q(T)}$, for some transcendental number T.

2 Main result

Proposition. *Fix non-constant polynomials $P(x)$, $Q(x) \in \mathbb{Q}[x]$. Then the set of algebraic numbers of the form $P(T)^{Q(T)}$, with T transcendental, is dense in some connected subset either of $\mathbb{R}$ or $\mathbb{C}$.*

As we said in Section 1, all algebraic numbers $A \geq e^{-1/e}$ satisfying $A^n \notin \mathbb{Q}$ for all $n \geq 1$, can be written in the form T^T, for some $T \notin \overline{\mathbb{Q}}$. An example of such A is $1+\sqrt{2}$. But that is only one case of our proposition, namely when $P(x) = Q(x) = x$. So for proving our result we need a stronger condition satisfied by an algebraic number A, and that is exactly what our next result asserts.

Lemma. *Let $Q(x)$ be a polynomial in $\mathbb{Q}[x]$ and set $\mathcal{F} = \{Q(x) - d : d \in \mathbb{Q}\}$. Then there exists $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}}$, such that*

$$\alpha^n \notin \mathbb{Q}(\mathcal{R}_{\mathcal{F}}) \ \ \textit{for all} \ \ n \geq 1, \tag{1}$$

where $\mathcal{R}_{\mathcal{F}}$ denotes the set $\{x \in \mathbb{C} : f(x) = 0$ for some $f \in \mathcal{F}\}$.

Proof. Set $\mathcal{F} = \{F_1, F_2, \ldots\}$, and for each $n \geq 1$, set $K_n = \mathbb{Q}(\mathcal{R}_{F_1 \ldots F_n})$ and $[K_n : \mathbb{Q}] = t_n$. Since $K_n \subseteq K_{n+1}$, then $t_n | t_{n+1}$, for all $n \geq 1$. Therefore, there are integers $(m_n)_{n \geq 1}$ such that $t_n = m_{n-1} \ldots m_1 t_1$. Note that $K_{n+1} = K_n(\mathcal{R}_{F_{n+1}})$ and $\deg F_{n+1} = \deg Q$. It follows that $[K_{n+1} : K_n] \leq (\deg Q)!$. Because $\mathbb{Q} \subseteq K_n \subseteq K_{n+1}$, we also have that $\frac{t_{n+1}}{t_n} \leq (\deg Q)!$ for all $n \geq 1$. On the other hand $\frac{t_{n+1}}{t_n} = m_n$, so the sequence $(m_n)_{n \geq 1}$ is bounded. Thus, we ensure the existence of a prime number $p > \max_{n \geq 1}\{m_n, t_1, 3\}$. Hence p does not divide t_n, for $n \geq 1$. We pick a real number α that is a root of the irreducible polynomial $F(x) = x^p - 4x + 2$ and we claim that $\alpha \notin \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$. Indeed, if this is not the case, then there exists a number $s \geq 1$, such that $\alpha \in \mathbb{Q}(\mathcal{R}_{F_1 \ldots F_s}) = K_s$. Since $[\mathbb{Q}(\alpha) : \mathbb{Q}] = p$, we would have that $p | t_s$, however this is impossible. Moreover, given $n \geq 1$, we have the field inclusions $\mathbb{Q} \subseteq \mathbb{Q}(\alpha^n) \subseteq \mathbb{Q}(\alpha)$. So $[\mathbb{Q}(\alpha^n) : \mathbb{Q}] = 1$ or p, but α^n cannot be written as radicals over $\mathbb{Q}$, since that $F(x)$ is not solvable by radicals over $\mathbb{Q}$, see [1, p. 189]. Hence $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^n)$ and then such α satisfies the condition (1). □

Without referring to the lemma, we have the following special remarks:

Remark 1 If $\deg Q(x) = 1$, then $\mathbb{Q}(\mathcal{R}_{\mathcal{F}}) = \mathbb{Q}$. Therefore $\alpha = 1 + \sqrt{2}$ satisfies our desired condition (1).

Remark 2 More generally, if $\deg Q \leq 4$, then we take α one of the real roots of the polynomial $F(x) = x^5 - 4x + 2$. We assert that this α satisfies (1). In fact, note that all elements of the field $\mathbb{Q}(\mathcal{R}_{\mathcal{F}})$ are solvable by radicals (over $\mathbb{Q}$), on the other hand the Galois group of $F(x) = 0$ over $\mathbb{Q}$ is isomorphic to S_5 (the symmetric group), see [1, p. 189]. Hence if $\alpha^n \in \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$, it would be expressed as radicals over $\mathbb{Q}$, but this cannot happen.

Now we are able to prove our main result:

Proof of the proposition. Let us suppose that P assumes a positive value. In this case, we have $0 < P(x) \neq 1$ for some interval $(a, b) \subseteq \mathbb{R}$. Therefore, the function $f : (a, b) \to \mathbb{R}$, given by $f(x) := P(x)^{Q(x)}$ is well-defined. Since f is a non-constant continuous function, $f((a, b))$ is a non-degenerate interval, say (c, d). Now, take α as in the lemma. Note that the set $\{\alpha Q : Q \in \mathbb{Q} \setminus \{0\}\}$ is dense in (c, d). For such an $\alpha Q \in (c, d)$, we have

$$\alpha Q = P(T)^{Q(T)} \tag{2}$$

for some $T \in (a, b)$. We must prove that T is a transcendental number. Assuming the contrary, then $P(T)$ and $Q(T)$ are algebraic numbers. Since $P(T) \notin \{0, 1\}$, then by the Gelfond-Schneider theorem, we infer that $Q(T) = \frac{r}{s} \in \mathbb{Q}$, $s > 0$. It follows that $T \in \mathcal{R}_{Q(x)-\frac{r}{s}} \subseteq \mathcal{R}_{\mathcal{F}}$, so $P(T)^r \in \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$. By (2), $(\alpha Q)^s = P(T)^r$, hence $\alpha^s \in \mathbb{Q}(\mathcal{R}_{\mathcal{F}})$, but that contradicts the lemma.

For the case that $P(x) \leq 0$ for all $x \in \mathbb{R}$, we can consider a subinterval $(a, b) \subseteq \mathbb{R}$ such that $\mathcal{R}_P \cap (a, b) = \emptyset$, therefore the proof follows by the same argument. But in this case the image of (a, b) under f is a connected subset of $\mathbb{C}$ and our basic dense subset (in $\mathbb{C}$) is the set $\{\alpha Q : Q \in \mathbb{Q}(i) \setminus \{0\}\}$. □

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Diego Marques
Departamento de Matemática
Universidade de Brasília
Brasília, DF, Brazil
e-mail: diego@mat.unb.br