Elemente der Mathematik

Two more triangle centers

Mowaffaq Hajja and Panagiotis T. Krasopoulos

Mowaffaq Hajja received his Ph.D. at Purdue University (Indiana, USA) in 1978. Since then he holds a position at Yarmouk University in Irbid (Jordan) where he is currently a professor. His mathematical interests include field theory and Euclidean geometry.

Panagiotis T. Krasopoulos received his Ph.D. in Engineering from the National Technical University of Athens (NTUA), Greece, in 2001. Currently, he is working at the Social Insurance Institute as a computer scientist. His mathematical interests include geometry, dynamical systems, and optimization.

1 Introduction

This note grew out of a question that a student asked while the first-named author was giving a lecture on triangle centers in a Geometry class in the fall of 2008. The student, Shefa'a Bani Melhem, wondered about the point in a given triangle that, when joined to the midpoints of the sides, divides the triangle into three quadrilaterals of equal area. Few days later, the aforementioned author asked her to prove that such a point is necessarily the centroid and to consider the point whose *perpendiculars* to the sides divide the triangle into three quadrilaterals of equal area. This problem turned out to be more difficult than was expected, and it is the purpose of this note to investigate the existence and uniqueness of such a point and to describe its trilinear coordinates (or simply, its trilinears), i.e., its

In dem nachfolgenden Beitrag untersuchen die Autoren die naheliegende Fragestellung, ob es in einem spitzwinkligen Dreieck ABC einen Punkt $\mathcal{E}$ mit der Eigenschaft gibt, dass die drei Vierecke, die durch das Fällen der Lote von $\mathcal{E}$ auf die Dreiecksseiten entstehen, flächengleich sind. Die Autoren beweisen, dass es in einem spitzwinkligen Dreieck genau einen solchen Punkt $\mathcal{E}$ gibt. Sie bestimmen auch die Gleichungen, denen die Abstände von $\mathcal{E}$ zu den Dreiecksseiten genügen. Im allgemeinen ist der Punkt $\mathcal{E}$ vom Umkreis- und Innkreismittelpunkt sowie vom Schwerpunkt und vom Höhenschnittpunkt verschieden. Fällt der Punkt $\mathcal{E}$ aber mit einem dieser klassischen Zentren zusammen, so weisen die Autoren nach, dass ein gleichseitiges Dreieck vorliegt.

directed distances to the sides of the triangle. The question whether this center can coincide with any of the traditional centers for a non-equilateral triangle is also addressed. We call such a point the *equiareality center* and we denote it by $\mathcal{E}$.

We show that $\mathcal{E}$ exists and is unique for acute triangles, and we write down equations that define its trilinears. However, these equations, though quite simple looking and elegant, are very hard to solve and everything suggests that this center is new. We also prove that this new center does not coincide with any of the traditional centers except when the triangle is equilateral.

In the course of investigation, a natural question arose and led to another center that we denoted by $\mathcal{E}_0$. Its properties and relation to other centers are also explored.

Many of the items in this paper can be used as projects, homeworks, examination problems, and issues for classroom discussion in a first course in Euclidean Geometry. In particular, dealing with the complexities that arise when one considers obtuse triangles is expected to generate fruitful discussions and to lead to interesting results.

2   Existence and uniqueness of $\mathcal{E}$ for acute triangles

Theorem 3 below establishes the existence and uniqueness of $\mathcal{E}$ for acute triangles. Lemma 1 is needed in its proof. In this lemma and throughout, the side lengths and angles of a triangle ABC are denoted by a, b, c, A, B, C, in the standard order, and the symbol $[\ldots]$ stands for the area.

Lemma 1. *Let ABC be an acute triangle and let X be a point on the side BC. Let XY, XZ be the perpendiculars dropped from X onto the sides AC, AB, respectively.*

  (i) *If $[XYC] \geq \frac{[ABC]}{4}$, then $[XZB] < \frac{[ABC]}{4}$.*
  (ii) *If $A \leq B$ and $A \leq C$, then $\frac{[XZB]+[XYC]}{[ABC]} \leq \frac{1}{2}$.*

Proof. (i) Suppose that $[XYC] \geq [ABC]/4$. Let $XC = t$ and let BS, CT be the perpendiculars dropped from B, C on AC, AB, respectively; see Fig. 1. Then

$$\frac{1}{4} \leq \frac{[XYC]}{[ABC]} < \frac{[XYC]}{[SBC]} = \frac{t^2}{a^2}.$$

Therefore $(t/a)^2 > 1/4$ and hence $t/a > 1/2$ and $(a-t)/a < 1/2$. Therefore

$$\frac{[XZB]}{[ABC]} < \frac{[XZB]}{[TBC]} = \left(\frac{a-t}{a}\right)^2 < \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

(ii) Let t, S, T be as in (i) and refer to Fig. 1 again. Suppose that $A \leq B$, $A \leq C$. Then $[ABC] \geq 2[TBC]$, $[ABC] \geq 2[SBC]$. It follows that

$$\frac{[XZB]}{[ABC]} + \frac{[XYC]}{[ABC]} \leq \frac{[XZB]}{2[TBC]} + \frac{[XYC]}{2[SBC]} = \frac{t^2}{2a^2} + \frac{(a-t)^2}{2a^2} \leq \frac{a^2}{2a^2} = \frac{1}{2},$$

as desired. $\square$
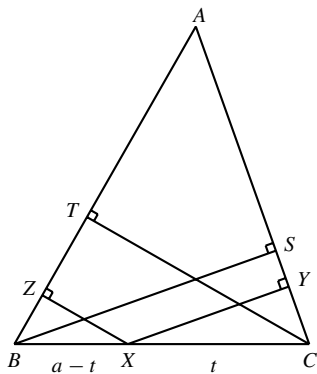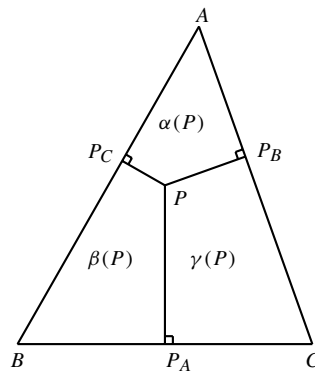
Fig. 1

Fig. 2

We remark that an interesting elaboration on Lemma 1(ii) can be found in [3].

Before proving the main theorem, we introduce notations and definitions that we shall use throughout.

Definition 2. For any acute triangle ABC, and any point P inside (or on the boundary of) ABC, we denote by P_A, P_B, P_C the orthogonal projections of P on the sides BC, CA, AB, respectively; see Fig. 2. We define $\alpha(P)$, $\beta(P)$, $\gamma(P)$ by

$$\alpha(P) = \frac{[AP_C P P_B]}{[ABC]}, \quad \beta(P) = \frac{[BP_A P P_C]}{[ABC]}, \quad \gamma(P) = \frac{[CP_B P P_A]}{[ABC]}. \tag{1}$$

Note that if P is on the boundary, then some of these quadrilaterals degenerate into triangles.

Theorem 3. *For any acute triangle, there exists a unique point whose perpendiculars to the sides divide the triangle into three quadrilaterals of equal area.*

Proof. Let ABC be an acute triangle, and assume without loss of generality that $A \leq B \leq C$.

Let P be any point on the line segment BC; see Fig. 3. As a point S moves from A to P, $\alpha(S)$ increases from $\alpha(A) = 0$ to $\alpha(P)$. By Lemma 1(ii), $\alpha(P) \geq 1/2$. Therefore, there exists a unique point on AP for which $\alpha = 1/3$. We denote this point by P^*. Thus for every P on BC, P^* is the point on AP for which $\alpha(P^*) = 1/3$. In particular, B^* is the point on AB whose perpendicular B^*U to AC has the property that $\alpha(B^*) = [B^*UA]/[ABC] = 1/3$; see Fig. 4. Similarly, C^* is the point on AC whose perpendicular C^*V to AB has the property that $\alpha(C^*) = [C^*VA]/[ABC] = 1/3$.

Since $\alpha(B^*) = 1/3 \geq 1/4$, Lemma 1(i) implies that $\beta(B^*) < 1/4 < 1/3$. Similarly, $\gamma(C^*) < 1/3$. Since $\alpha(B^*) = \alpha(C^*) = 1/3$ and $\alpha + \beta + \gamma = 1$, it follows that

$$\beta(B^*) < \frac{1}{3}, \quad \beta(C^*) > \frac{1}{3}. \tag{2}$$
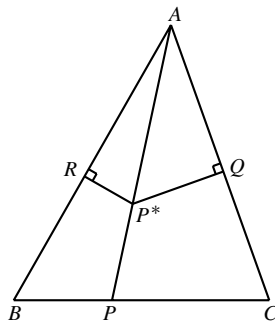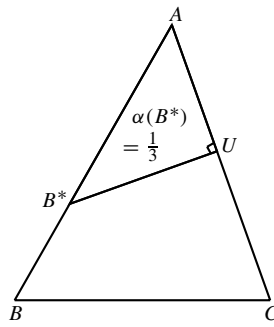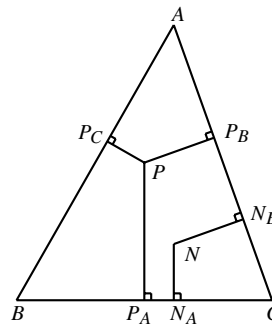
Fig. 3   Fig. 4   Fig. 5

By continuity of the function $P \mapsto \beta(P^*)$, there exists a point M on BC such that $\beta(M^*) = 1/3$. Since $\alpha(M^*) = 1/3$, it follows that $\alpha(M^*) = \beta(M^*) = \gamma(M^*) = 1/3$, and hence M^* has the required property.

To prove uniqueness, let P, N be two points with the given property. Then N must lie in one of the quadrilaterals determined by P, say N lies in $CP_B P P_A$; see Fig. 5. Then $\gamma(N) \leq \gamma(P)$ with equality if and only if $N = P$. Since $\gamma(N) = \gamma(P) = 1/3$, it follows that $N = P$.   □

Definition 4. For an acute triangle ABC, the point P for which

$$\alpha(P) = \beta(P) = \gamma(P) \tag{3}$$

is called the *equiareality center* of ABC and is denoted by $\mathcal{E}$.

3   Trilinears of $\mathcal{E}$

Let ABC be an acute triangle and let x, y, z be the trilinears of the equiareality center $\mathcal{E}$. Drop perpendiculars $\mathcal{E}Y$, $\mathcal{E}Z$ onto the sides AC, AB, respectively. Let $A = 2\xi$, $B = 2\eta$, $C = 2\zeta$. Let $\angle BA\mathcal{E} = \xi - p$, $\angle CA\mathcal{E} = \xi + p$; see Fig. 6. Then

$$\frac{z}{y} = \frac{\sin(\xi - p)}{\sin(\xi + p)} = \frac{\sin\xi\cos p - \cos\xi\sin p}{\sin\xi\cos p + \cos\xi\sin p} = \frac{1 - \cot\xi\tan p}{1 + \cot\xi\tan p},$$

$$\cot\xi\tan p = \frac{y - z}{y + z},$$
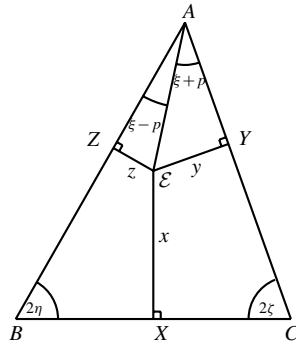
$$\tan p = \frac{y - z}{y + z}\tan\xi.$$

Fig. 6

$$
\begin{aligned}
2[AZ\mathcal{E}Y] &= yz\ \sin(2\xi) + y\cot(\xi + p)\, z\cot(\xi - p)\sin 2\xi \\
&= yz\ \sin(2\xi)(1 + \cot(\xi + p)\cot(\xi - p)) \\
&= yz\ \sin(2\xi)\ \left(1 + \frac{1 - \tan\xi \tan p}{\tan\xi + \tan p}\,\frac{1 + \tan\xi \tan p}{\tan\xi - \tan p}\right) \\
&= yz\ \sin(2\xi)\ \frac{(1 + \tan^2\xi)(1 - \tan^2 p)}{\tan^2\xi - \tan^2 p} \\
&= 2yz\ \sin\xi \cos\xi\ \frac{\sec^2\xi(1 - \tan^2 p)}{\tan^2\xi - \tan^2 p} \\
&= 2yz\ \tan\xi\ \frac{1 - \tan^2 p}{\tan^2\xi - \tan^2 p} \\
&= 2yz\ \tan\xi\ \frac{(y + z)^2(1 - \tan^2 p)}{(y + z)^2 \tan^2\xi - (y - z)^2 \tan^2\xi} \\
&= \frac{(y + z)^2 - (y - z)^2 \tan^2\xi}{2\tan\xi}.
\end{aligned}
$$

We may assume that $[ABC] = 3/2$. Then we have

$$
(y + z)^2 - (y - z)^2 \tan^2\xi = 2\tan\xi,
$$
$$
(1 - \tan^2\xi)(y^2 + z^2) + 2yz(1 + \tan^2\xi) = 2\tan\xi,
$$
$$
(y^2 + z^2)\cos A + 2yz - \sin A = 0.
$$

We record this in the following theorem.

Theorem 5. *The trilinears $x : y : z$ of the equiareality center $\mathcal{E}$ of an acute triangle ABC are defined by the equations*

$$
\left.
\begin{aligned}
(y^2 + z^2)\cos A + 2yz - \sin A &= 0, \\
(z^2 + x^2)\cos B + 2zx - \sin B &= 0, \\
(x^2 + y^2)\cos C + 2xy - \sin C &= 0.
\end{aligned}
\right\}
\tag{4}
$$

4   Coincidence of $\mathcal{E}$ with the other traditional centers

Let ABC be a triangle and let $\mathcal{G}$, $\mathcal{O}$, $\mathcal{I}$, $\mathcal{H}$ denote its centroid, circumcenter, incenter, orthocenter, respectively. We shall show that $\mathcal{E}$ cannot coincide with any of these centers except when ABC is equilateral. For $\mathcal{E}$ to be defined, we assume of course that ABC is acute.

Note that to study the possible coincidence of $\mathcal{E}$ with a center whose trilinears (x_0, y_0, z_0) are given, one needs only substitute $(x, y, z) = \lambda(x_0, y_0, z_0)$ in (4) and then solves the resulting equations. This may turn out to be difficult and clumsy. In this case, a purely geometric treatment would be desirable. Also, it may turn out that the $(\alpha : \beta : \gamma)$-coordinates (as defined in (1)) of a given center P are easy to calculate. One then sets these equal to $(1 : 1 : 1)$ and solves the resulting equations. All of these approaches are illustrated below.

For the reader's convenience, we list below trilinears and barycentrics of the traditional centers. For these and anything that has to do with triangle centers, we refer the reader to [4] and [5].

Center	Trilinears	Barycentrics
Centroid $\mathcal{G}$	$\left(\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right)$	$(1 : 1 : 1)$
Circumcenter $\mathcal{O}$	$(\cos A : \cos B : \cos C)$	$(\sin 2A : \sin 2B : \sin 2C)$
Incenter $\mathcal{I}$	$(1 : 1 : 1)$	$(a : b : c) = (\sin A : \sin B : \sin C)$
Orthocenter $\mathcal{H}$	$(\sec A : \sec B : \sec C)$	$(\tan A : \tan B : \tan C)$

In the next theorems, we refer to (1) for the definitions of α, β, γ.

Theorem 6. *If ABC is an acute triangle in which $\mathcal{E} = \mathcal{I}$ or $\mathcal{E} = \mathcal{O}$, then ABC is equilateral.*

Proof. Using the facts that

$$\left(\alpha(\mathcal{I}) : \beta(\mathcal{I}) : \gamma(\mathcal{I})\right) = \left(\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2}\right),$$
$$\left(\alpha(\mathcal{O}) : \beta(\mathcal{O}) : \gamma(\mathcal{O})\right) = (\sin 2B + \sin 2C : \sin 2A + \sin 2C : \sin 2A + \sin 2B),$$
$$\left(\alpha(\mathcal{E}) : \beta(\mathcal{E}) : \gamma(\mathcal{E})\right) = (1 : 1 : 1),$$

we conclude that

$$\mathcal{I} = \mathcal{E} \implies \cot\frac{A}{2} = \cot\frac{B}{2} = \cot\frac{C}{2} \implies A = B = C,$$
$$\mathcal{O} = \mathcal{E} \implies \sin 2B + \sin 2C = \sin 2C + \sin 2A = \sin 2A + \sin 2B$$
$$\implies \sin 2A = \sin 2B = \sin 2C \implies A = B = C.$$

The last implication follows from the fact that the possibility $2B + 2C = \pi$ leads to $A = \pi/2$, contradicting the assumption that ABC is acute. This will be freely used later. $\square$

Theorem 7. *If ABC is an acute triangle in which $\mathcal{E} = \mathcal{G}$, then ABC is equilateral.*

Proof. It is obvious that the medians AA', BB', CC' of ABC divide ABC into six triangles of equal areas; see [1, Theorem 5]. Therefore the quadrilaterals $\mathcal{G}C'AB'$, $\mathcal{G}A'BC'$, $\mathcal{G}B'CA'$ have equal areas. Also, the orthocenter $\mathcal{H}$ of ABC must lie in one of these quadrilaterals, say in $\mathcal{G}B'CA'$; see Fig. 7. Then

$$\angle BB'C \le \frac{\pi}{2}, \quad \angle AA'C \le \frac{\pi}{2}. \tag{5}$$

Hence the orthogonal projections J, I of $\mathcal{G}$ onto the sides CA, CB, respectively, lie in the quadrilateral $\mathcal{G}B'CA'$, and thus $\mathcal{G}ICJ$ lies inside $\mathcal{G}B'CA'$. If $\mathcal{E} = \mathcal{G}$, then these quadrilaterals have the same area and hence $B' = J$ and $A' = I$. Thus the medians BB', AA' are perpendicular to the respective sides and the triangle is equilateral, as claimed. $\square$
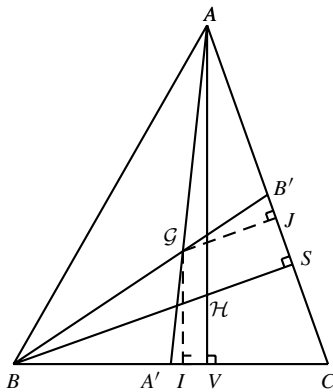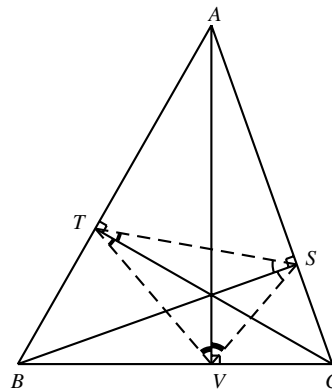

Fig. 7   Fig. 8

Theorem 8. *If ABC is an acute triangle in which $\mathcal{E} = \mathcal{H}$, then ABC is equilateral.*

Proof. Let AV, BS, CT be the altitudes of ABC (necessarily through $\mathcal{H}$); see Fig. 8. Clearly the triangles CSV and CBA are similar with similarity ratio $CS/CB = \cos C$. Therefore $[CSV]/[CBA] = \cos^2 C$. Similarly, $[BVT]/[CBA] = \cos^2 B$. Hence

$$\frac{[CSV]}{[BVT]} = \frac{\cos^2 C}{\cos^2 B}. \tag{6}$$

Also, it follows from the cyclicity of the quadrilaterals $AT\mathcal{H}S$, $BV\mathcal{H}T$, $CS\mathcal{H}V$ that the angles of VST are given by

$$V = \pi - 2A, \quad S = \pi - 2B, \quad T = \pi - 2C, \tag{7}$$

and that AV, BS, CT are the angle bisectors of the triangle VST. Therefore

$$\frac{[VS\mathcal{H}]}{[VT\mathcal{H}]} = \frac{(VS)(V\mathcal{H}) \sin \angle \mathcal{H}VS}{(VT)(V\mathcal{H}) \sin \angle \mathcal{H}VT} = \frac{VS}{VT} = \frac{\sin T}{\sin S} = \frac{\sin(\pi - 2C)}{\sin(\pi - 2B)} = \frac{\sin 2C}{\sin 2B}. \tag{8}$$

Note also that

$$\sin 2C - \sin 2B = 2\cos(C+B)\sin(C-B) = -2\cos A\sin(C-B). \qquad (9)$$

Suppose now that $C > B$. Then it follows from (8) and (9) that $[VS\mathcal{H}] < [VT\mathcal{H}]$. It also follows from (6) that $[CSV] < [BVT]$. Adding, we obtain $[CS\mathcal{H}V] < [BV\mathcal{H}T]$. Thus we have proved that

$$C > B \Longrightarrow \gamma(\mathcal{H}) < \beta(\mathcal{H}). \qquad (10)$$

Suppose now that $\mathcal{E} = \mathcal{H}$. Then $\gamma(\mathcal{H}) = \beta(\mathcal{H})$ and hence $C = B$. Similarly $B = A$ and ABC is equilateral. $\qquad \square$

Remark 9. It is well-known that if any two of the centers $\mathcal{G}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{H}$ of a triangle coincide, then the triangle is equilateral; see [2] and the references therein. It also goes without saying that all centers of an equilateral triangle coincide.

In view of Remark 9 above, Theorems 6, 7, 8 can be summarized in the next theorem.

Theorem 10. *An acute triangle is equilateral if and only if two (and hence all) of the centers $\mathcal{E}$, $\mathcal{G}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{H}$ coincide.*

5 Another related center

Let ABC be an acute triangle. For X on BC, let XY, XZ be the perpendiculars from X dropped onto the sides AC, AB, respectively. As X moves from B to C, $[XZB]/[XYC]$ increases from 0 to ∞. Thus there exists a unique point, to be denoted by A', for which $[XZB] = [XYC]$. We define B', C' similarly.

Theorem 11. *The cevians AA', BB', CC' are concurrent. The barycentrics of the point of concurrence are given by*

$$\sqrt{\sin 2A} : \sqrt{\sin 2B} : \sqrt{\sin 2C}, \qquad (11)$$

and the trilinears are given by

$$\sqrt{\cot A} : \sqrt{\cot B} : \sqrt{\cot C}. \qquad (12)$$

Proof. Observe that if X lies on BC and if $t = CA'$, as shown in Fig. 1, then the condition $[XZB] = [XYC]$ is equivalent to $(a-t)^2 \cos B \sin B = t^2 \cos C \sin C$, i.e., $(a-t)^2 : t^2 = \sin 2C : \sin 2B$. Concurrence of AA', BB', CC' follows immediately from Ceva's theorem. It also follows that if P is the point of concurrence, then $[PAB] : [PAC] = (a-t) : t = \sqrt{\sin 2C} : \sqrt{\sin 2B}$. Hence the barycentrics of P are as given in (11). The statement about the trilinears follows from

$$\frac{\sqrt{\sin 2A}}{a} \sim \frac{\sqrt{\sin 2A}}{\sin A} = \sqrt{\frac{2\sin A\cos A}{\sin^2 A}} \sim \sqrt{\cot A}. \qquad \square$$

Definition 12. The point of concurrence will be denoted by $\mathcal{E}_0$.

Theorem 13. *An acute triangle is equilateral if and only if two* (*and hence all*) *of the centers $\mathcal{E}$, $\mathcal{E}_0$, $\mathcal{G}$, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{H}$ coincide.*

Proof. In view of Remark 9 above and in view of the symmetry, it is sufficient to show that each of the assumptions $\mathcal{E}_0 = \mathcal{G}$, $\mathcal{E}_0 = \mathcal{O}$, $\mathcal{E}_0 = \mathcal{I}$, $\mathcal{E}_0 = \mathcal{H}$, $\mathcal{E}_0 = \mathcal{E}$ leads to $A = B$. But

$$\mathcal{E}_0 = \mathcal{G} \implies \sqrt{\sin 2A} = \sqrt{\sin 2B} \implies A = B,$$

$$\mathcal{E}_0 = \mathcal{O} \implies \frac{\sin 2A}{\sqrt{\sin 2A}} = \frac{\sin 2B}{\sqrt{\sin 2B}} \implies \sin 2A = \sin 2B \implies A = B,$$

$$\mathcal{E}_0 = \mathcal{I} \implies \frac{\sin A}{\sqrt{\sin 2A}} = \frac{\sin B}{\sqrt{\sin 2B}} \implies \tan A = \tan B \implies A = B,$$

$$\mathcal{E}_0 = \mathcal{H} \implies \frac{\tan A}{\sqrt{\sin 2A}} = \frac{\tan B}{\sqrt{\sin 2B}} \implies \frac{\sin A}{\cos^3 A} = \frac{\sin B}{\cos^3 B}$$

$$\implies \tan A \left(1 + \tan^2 A\right) = \tan B \left(1 + \tan^2 B\right)$$

$$\implies A = B, \text{ because } x(1 + x^2) \text{ is increasing.}$$

It remains to deal with the case $\mathcal{E} = \mathcal{E}_0$. If $\mathcal{E} = \mathcal{E}_0$, then a multiple of the trilinears of $\mathcal{E}_0$ given in (12) must satisfy the equations (4) for the trilinears of $\mathcal{E}$. Substituting $(x, y, z) = \lambda \left(\sqrt{\cot A}, \sqrt{\cot B}, \sqrt{\cot C}\right)$ in the first equation in (4), we obtain

$$\lambda^2 \left(\frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}\right) \cos A + 2\lambda^2 \sqrt{\frac{\cos B \cos C}{\sin B \sin C}} = \sin A.$$

Multiplying by $(\sin B \sin C)/\lambda^2$ and using the identity $\cos B \sin C + \sin B \cos C = \sin(B + C) = \sin A$, we obtain

$$\sin A \cos A + 2\sqrt{\sin B \cos B \sin C \cos C} = \frac{\sin A \sin B \sin C}{\lambda^2}.$$

Using the double angle formula, we obtain

$$\sin 2A + 2\sqrt{\sin 2B \sin 2C} = \frac{2 \sin A \sin B \sin C}{\lambda^2}.$$

Using the other two equations in (4), and letting

$$u = \sqrt{\sin 2A}, \quad v = \sqrt{\sin 2B}, \quad w = \sqrt{\sin 2C},$$

we obtain

$$u^2 + 2vw = v^2 + 2wu = w^2 + 2uv.$$

Therefore

$$(u - v)(u + v - 2w) = (v - w)(v + w - 2u) = (w - u)(w + u - 2v) = 0.$$

If no two of u, v, w are equal, then $u + v - 2w = v + w - 2u = 0$, and hence $u + v + w = 3w = 3u$, and the contradiction $w = u$. If exactly two of u, v, w are equal, say $u = v \neq w$, then $v + w - 2u = 0$ and hence the contradiction $w = u$. We are left with the possibility $u = v = w$, i.e., $\sin 2A = \sin 2B = \sin 2C$. Therefore $A = B = C$. □

6 The obtuse case

If ABC is an obtuse triangle and if P is a point inside ABC, then the perpendiculars PX, PY, PZ from P to the sides BC, CA, AB do not necessarily fall onto the sides; see Fig. 9. In this case, parts of the quadrilaterals $PYAZ$, $PZBX$, $PXCY$ fall outside the triangle, and some of these quadrilaterals are self crossing; see Fig. 9. It would be interesting to
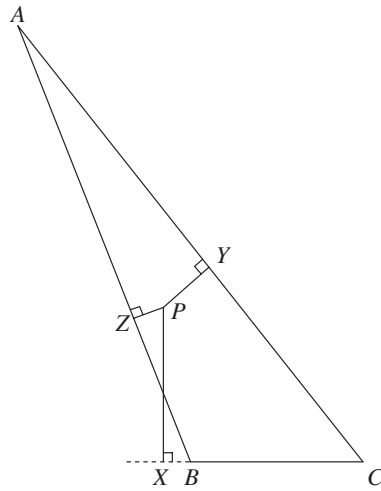

Fig. 9

explore possibilities of assigning areas to such quadrilaterals in such a way that the sum of the areas of $PYAZ$, $PZBX$, $PXCY$ is still equal to that of ABC. One would also like to do the same for points outside ABC. Then one would investigate the existence and uniqueness of a point P, inside or outside ABC, for which the areas of $PYAZ$, $PZBX$, $PXCY$ are equal. Calling such a point an equiareality point, one may try to characterize those obtuse triangles that have an interior equiareality point, and those obtuse triangles that have an interior equiareality point P for which the three associated quadrilaterals lie inside ABC.

References

[1] Abu-Saymeh, S.; Hajja, M.: In search of more triangle centers – A source of classroom projects in Euclidean geometry. *Internat. J. Math. Ed. Sci. Tech.* 36 (2005), 889–912.

[2] Hajja, M.: Triangle centers – Some questions in Euclidean geometry. *Internat. J. Math. Ed. Sci. Tech.* 32 (2001), 21–37.

[3] Hajja, M.; Krasopoulos, P.T.: Problem Proposal. Aufgabe 1275, *El. Math.* 65, No. 1 (2010), 37.

[4] Kimberling, C.: Triangle centers and central triangles. *Congr. Numer.* 129 (1998), 1–285.

[5] Kimberling, C.: Encyclopedia of Triangle Centers. http://faculty.evansville.edu/ck6/encyclopedia/

Mowaffaq Hajja
Mathematics Department
Yarmouk University
Irbid, Jordan
e-mail: `mhajja@yu.edu.jo`
        `mowhajja@yahoo.com`

Panagiotis T. Krasopoulos
Social Insurance Institute
General Administration of Informatics
Papadiamantopoulou 87
115 27 Athens, Greece
e-mail: `pankras@in.gr`