

## The spectral radius in $\mathcal{C}_0(X)$ -Banach algebras

Walther Paravicini

**Abstract.** The spectral radius of an element of a  $\mathcal{C}_0(X)$ -Banach algebra can be calculated as the supremum of the spectral radii in the fibres. As a consequence, calculations in K-theory for such algebras can potentially be carried out fibrewise.

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### 1. Introduction

The impetus to write this article has been to give a proof of the fact that the Banach algebras  $\mathcal{A}(G, \mathcal{C}_0(\mathbb{R}, B))$  and  $\mathcal{C}_0(\mathbb{R}, \mathcal{A}(G, B))$  are isomorphic in (topological) K-theory, where  $G$  is a locally compact Hausdorff group,  $B$  is a  $G$ -Banach algebra and  $\mathcal{A}(G)$  is an unconditional completion of  $\mathcal{C}_c(G)$ ; as an example,  $L^1(G, \mathcal{C}_0(\mathbb{R}, B))$  and  $\mathcal{C}_0(\mathbb{R}, L^1(G, B))$  are isomorphic in K-theory. These results already appeared in the work of V. Lafforgue, see [Laf02], Section 1.7, and also a variant for groupoids can be found in [Laf06]. Results of this type are important because they make it possible to lift positive results on the Bost conjecture for  $K_0$  to higher topological K-groups without any extra work. However, no proof has been published yet.

In the present article, the original, direct argument that V. Lafforgue has indicated to me is generalised in several directions. Firstly, we replace  $\mathbb{R}$  by a general second countable locally compact Hausdorff space  $X$ . The main part of the argument is now a statement about  $\mathcal{C}_0(X)$ -Banach algebras, namely the spectral radius formula mentioned in the abstract (see Theorem 3.4) and its consequence

**Theorem 1.1.** *Let  $A$  and  $A'$  be  $\mathcal{C}_0(X)$ -Banach algebras and let  $\varphi: A \rightarrow A'$  be a  $\mathcal{C}_0(X)$ -linear contractive homomorphism of Banach algebras. Assume that  $\varphi_x: A_x \rightarrow A'_x$  is spectral and dense for all  $x \in X$  and that  $X$  is second countable. Then  $\varphi$  is spectral and dense and, therefore,  $\varphi_*: K_*(A) \rightarrow K_*(A')$  is an isomorphism.*

Recall that a continuous homomorphism  $\psi: B \rightarrow B'$  of Banach algebras is called spectral if it preserves the spectral radius of elements, or, equivalently, if  $b \in B$  is

invertible in  $B$  if and only if  $\psi(b)$  is invertible in  $B'$ ; the morphism  $\psi$  is called dense if its image  $\psi(B)$  is dense in  $B'$ . It is a well-known fact that spectral and dense homomorphisms are isomorphisms in  $K$ -theory, see for example [CMR07] or [Bos90] (and the references therein) and the survey article [Nic10] for more information on spectral and dense morphisms and their relation to  $K$ -theory.

To complete the proof of the original result on unconditional completions, what is now sufficient to show is that the fibres of  $\mathcal{A}(G, \mathcal{C}_0(X, B))$  and  $\mathcal{C}_0(X, \mathcal{A}(G, B))$  are isometrically isomorphic (because isomorphisms are clearly spectral and dense). It turns out that this is, at the heart of it, a statement not about Banach algebras, but about Banach spaces, and not about groups but about locally compact spaces.

If we replace the group  $G$  (or, more generally, the groupoid  $\mathcal{G}$ ) with an arbitrary locally compact Hausdorff space  $Y$  and the unconditional completion  $\mathcal{A}(G)$  with what we call a monotone completion  $\mathcal{H}(Y)$  (which is, technically, the same as an unconditional completion but without the algebra structure), and if  $E = (E_y)_{y \in Y}$  is an upper semi-continuous (u.s.c.) field of Banach spaces over  $Y$ , then the result we are going to show can now be formulated as follows:

**Theorem 1.2.** *The two  $\mathcal{C}_0(X)$ -Banach spaces  $\mathcal{H}(Y, EX)$  and  $\mathcal{H}(Y, E)X$  have the same fibres over points in  $X$ .*

To make this statement precise and to prove it will be the subject of Section 4.2. But before we start, we list some consequences of our main theorems.

**Corollary 1.3.** *Let  $G$  be a locally compact Hausdorff group and let  $B$  be a  $G$ -Banach algebra. Let  $\mathcal{A}(G)$  be an unconditional completion of  $\mathcal{C}_c(G)$ . Let  $X$  be a locally compact Hausdorff space. Then the canonical homomorphism of  $\mathcal{C}_0(X)$ -Banach algebras*

$$\iota: \mathcal{A}(G, BX) \rightarrow \mathcal{A}(G, B)X$$

*is an isometric isomorphism on the fibres. If  $X$  is second countable, then this means that  $\iota$  is spectral and dense, and hence*

$$\iota_*: K_*(\mathcal{A}(G, BX)) \cong K_*(\mathcal{A}(G, B)X).$$

*Proof.* Take  $(G, X, \mathcal{A}, B_G)$  instead of  $(Y, X, \mathcal{H}, E)$  in Theorem 1.2. Here  $B_G$  denotes the trivial field of Banach spaces over  $G$  with fibre  $B$ .  $\square$

**Corollary 1.4.** *In particular, taking  $X = \mathbb{R}$  in the preceding corollary, we obtain an isomorphism*

$$K_*(\mathcal{A}(G, SB)) \cong K_*(S\mathcal{A}(G, B)) \cong K_{*+1}(\mathcal{A}(G, B)),$$

*where  $S$  denotes the suspension functor for Banach algebras  $A \mapsto \mathcal{C}_0(\mathbb{R}, A)$ .*

**Corollary 1.5.** *Let  $\mathcal{G}$  be a locally compact Hausdorff groupoid equipped with a Haar system and let  $B$  be a  $\mathcal{G}$ -Banach algebra. Let  $\mathcal{A}(\mathcal{G})$  be an unconditional completion of  $\mathcal{C}_c(\mathcal{G})$ . Let  $X$  be a locally compact Hausdorff space. Then the canonical homomorphism of  $\mathcal{C}_0(X)$ -Banach algebras*

$$\iota: \mathcal{A}(\mathcal{G}, BX) \rightarrow \mathcal{A}(\mathcal{G}, B)X$$

*is an isometric isomorphism on the fibres (note that  $BX$  is a  $\mathcal{G}$ -Banach algebra in a canonical fashion). If  $X$  is second countable, then  $\iota$  is spectral and dense, and hence*

$$\iota_*: K_*(\mathcal{A}(\mathcal{G}, BX)) \cong K_*(\mathcal{A}(\mathcal{G}, B)X).$$

*Proof.* Take  $(\mathcal{G}, X, \mathcal{A}, r^*B)$  instead of  $(Y, X, \mathcal{H}, E)$  in Theorem 1.2. □

**Corollary 1.6.** *We have an isomorphism*

$$K_*(\mathcal{A}(\mathcal{G}, SB)) \cong K_*(S\mathcal{A}(\mathcal{G}, B)) \cong K_{*+1}(\mathcal{A}(\mathcal{G}, B)).$$

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**Note.** All Banach spaces and Banach algebras that appear in this article are supposed to be complex.

## 2. $\mathcal{C}_0(X)$ -Banach algebras and their fibres

Let  $X$  be a locally compact Hausdorff space. The notion of a  $\mathcal{C}_0(X)$ - $C^*$ -algebra is well known in the literature, and it has already been generalised to the concept of a  $\mathcal{C}_0(X)$ -Banach algebra, see [Bla96]. Here, we define it by introducing first what a  $\mathcal{C}_0(X)$ -Banach space is:

A  $\mathcal{C}_0(X)$ -Banach space is by definition a non-degenerate Banach  $\mathcal{C}_0(X)$ -module, where non-degeneracy means that  $\mathcal{C}_0(X) \cdot E$  is (dense in)  $E$ . If  $E$  and  $F$  are  $\mathcal{C}_0(X)$ -Banach spaces, then we take the bounded linear  $\mathcal{C}_0(X)$ -linear maps from  $E$  to  $F$  as morphisms from  $E$  to  $F$ . We are going to denote the morphisms from  $E$  to  $F$  by  $L^{\mathcal{C}_0(X)}(E, F)$ .

If  $E$  is a Banach space, then  $EX = \mathcal{C}_0(X, E)$  is a  $\mathcal{C}_0(X)$ -Banach space with the canonical action of  $\mathcal{C}_0(X)$ .

**Definition 2.1.** A  $\mathcal{C}_0(X)$ -Banach algebra  $B$  is a Banach algebra  $B$  which is at the same time a  $\mathcal{C}_0(X)$ -Banach space such that the multiplication of  $B$  is  $\mathcal{C}_0(X)$ -bilinear.

A homomorphism of  $\mathcal{C}_0(X)$ -Banach algebras  $\varphi: A \rightarrow B$  is simply a continuous  $\mathcal{C}_0(X)$ -linear homomorphism  $\varphi$  of algebras.

Let  $E$  be an upper semi-continuous field of Banach spaces over  $X$  (see [Laf06]). Consider  $\Gamma_0(X, E)$ , the space of all sections of  $E$  which vanish at infinity. The Banach space  $\Gamma_0(X, E)$  carries a canonical action of  $\mathcal{C}_0(X)$  making it a  $\mathcal{C}_0(X)$ -Banach space.

Conversely:

**Definition 2.2.** Let  $E$  be a  $\mathcal{C}_0(X)$ -Banach space. For all  $x \in X$ , the quotient Banach space  $E_x = E/(\mathcal{C}_0(X \setminus \{x\})E)$  is called the *fibre* of  $E$  over  $x$ ; it comes with a natural quotient map  $E \ni e \mapsto e_x \in E_x$ .

One can regard  $\mathfrak{F}(E) := (E_x)_{x \in X}$  as an upper semi-continuous field of Banach spaces over  $X$ . Let us denote the  $\mathcal{C}_0(X)$ -Banach space  $\Gamma_0(X, \mathfrak{F}(E))$  by  $\mathfrak{G}(E)$  and call it the *Gelfand transform* of  $E$ . There is a canonical contractive linear map  $\iota_E$  from  $E$  to  $\mathfrak{G}(E)$  which sends every  $e \in E$  to the section  $x \mapsto e_x \in E_x$ . Sadly enough,  $\iota_E$  needs neither be injective nor surjective, we only know that it has dense image; we do not have  $E \cong \Gamma_0(X, \mathfrak{F}(E))$  in general. The linear map  $\iota_E$  is isometric (and therefore an isomorphism) if and only if the  $\mathcal{C}_0(X)$ -Banach space  $E$  is what is called *locally  $\mathcal{C}_0(X)$ -convex*, i.e., for all  $\chi_1, \chi_2 \in \mathcal{C}_0(X)$ ,  $\chi_1, \chi_2 \geq 0$ ,  $\chi_1 + \chi_2 \leq 1$  and all  $e_1, e_2 \in E$  we have

$$\|\chi_1 e_1 + \chi_2 e_2\| \leq \max\{\|e_1\|, \|e_2\|\};$$

see [Gie82] and also Appendix A.2 of [Par07].

If  $E$  is an upper semi-continuous field of Banach spaces over  $X$ , then  $\Gamma_0(X, E)$  is automatically locally  $\mathcal{C}_0(X)$ -convex. Actually,  $E \mapsto \Gamma_0(X, E)$  defines an equivalence of categories between the category of upper semi-continuous fields of Banach spaces over  $X$  and the category of locally  $\mathcal{C}_0(X)$ -convex  $\mathcal{C}_0(X)$ -Banach spaces, the inverse functor being  $\mathfrak{F}(\cdot)$ . Therefore the functor  $E \mapsto \mathfrak{G}(E)$  on the category of  $\mathcal{C}_0(X)$ -Banach spaces has its values in the subcategory of locally  $\mathcal{C}_0(X)$ -convex  $\mathcal{C}_0(X)$ -Banach spaces. It is a projector in the sense that  $\mathfrak{G}(\mathfrak{G}(E))$  is naturally isomorphic to  $\mathfrak{G}(E)$ .

The functors  $\mathfrak{F}(\cdot)$  and  $\mathfrak{G}(\cdot)$  can also be applied to  $\mathcal{C}_0(X)$ -Banach algebras; the fact that underlies this observation is that  $\mathfrak{F}(\cdot)$  and  $\mathfrak{G}(\cdot)$  are compatible with the (fibrewise) tensor product of upper semi-continuous fields of Banach spaces over  $X$  and the  $\mathcal{C}_0(X)$ -tensor product of  $\mathcal{C}_0(X)$ -Banach spaces; this can be proved using the result that the  $\mathcal{C}_0(X)$ -tensor product of locally  $\mathcal{C}_0(X)$ -convex spaces is again locally  $\mathcal{C}_0(X)$ -convex, see [Par08].

So if we have a  $\mathcal{C}_0(X)$ -Banach algebra  $A$ , then we can form an upper semi-continuous field  $\mathfrak{F}(A)$  of Banach algebras over  $X$  and a  $\mathcal{C}_0(X)$ -Banach algebra  $\mathfrak{G}(A) = \Gamma_0(X, \mathfrak{F}(A))$ , and a  $\mathcal{C}_0(X)$ -Banach algebra  $A$  is called locally  $\mathcal{C}_0(X)$ -convex if and only if the underlying  $\mathcal{C}_0(X)$ -Banach space is locally  $\mathcal{C}_0(X)$ -convex, and this is the case if and only if the canonical homomorphism  $\iota_A: A \rightarrow \mathfrak{G}(A)$  is an (isometric) isomorphism of Banach algebras.

### 3. The spectral radius in $\mathcal{C}_0(X)$ -Banach algebras

In this section, we analyse to what extent the spectral radius of an element of a  $\mathcal{C}_0(X)$ -Banach algebra is determined by its fibrewise spectral radii. If  $A$  is a Banach algebra, then we write  $\rho_A(a)$  for the spectral radius of  $a \in A$  in  $A$ .

**3.1. A formula for the spectral radius.** In this paragraph, let  $A$  be a  $\mathcal{C}_0(X)$ -Banach algebra.

**Lemma 3.1.** *For all  $x_0 \in X$ , for all  $a \in A$  and all  $\varepsilon > 0$  there is a neighbourhood  $V$  of  $x_0$  in  $X$  such that for all  $\chi \in \mathcal{C}_c(X)$  with  $0 \leq \chi \leq 1$  and  $\text{supp } \chi \subseteq V$  we have*

$$\|\chi a\| \leq \|a_{x_0}\| + \varepsilon.$$

*Proof.* Let  $x_0 \in X$ ,  $a \in A$  and  $\varepsilon > 0$ . Recall the following formula of Varela (see [Var74]; it also follows from Lemme 1.10 of [Bla96] or Lemma 4.2.6 of [Par07]):

$$\|a_{x_0}\| = \inf\{\|\chi a\| : \chi \in \mathcal{C}_c(X) \exists V \subseteq X \text{ open} : \chi|_V = 1, 0 \leq \chi \leq 1, x_0 \in V\}.$$

In particular, we can choose a  $\chi' \in \mathcal{C}_c(X)$  such that  $0 \leq \chi' \leq 1$  and  $\chi' \equiv 1$  on a neighbourhood  $V$  of  $x_0$  and such that  $\|\chi' a\| \leq \|a_{x_0}\| + \varepsilon$ . Let  $\chi \in \mathcal{C}_c(X)$  be such that  $0 \leq \chi \leq 1$  and  $\text{supp } \chi \subseteq V$ . Then  $\chi \chi' = \chi$  and hence

$$\|\chi a\| = \|\chi \chi' a\| \leq \|\chi\|_\infty \|\chi' a\| \leq \|a_{x_0}\| + \varepsilon. \quad \square$$

**Lemma 3.2.** *For all  $a \in A$ , all  $x_0 \in X$  and all  $\varepsilon > 0$  there exists a neighbourhood  $V$  of  $x_0$  such that for all  $\chi \in \mathcal{C}_c(X)$  with  $0 \leq \chi \leq 1$  and  $\text{supp } \chi \subseteq V$  we have*

$$\rho_A(\chi a) \leq \rho_{A_{x_0}}(a_{x_0}) + \varepsilon.$$

*Proof.* Let  $a \in A$ ,  $x_0 \in X$  and  $\varepsilon > 0$ . Find a  $k \in \mathbb{N}$  such that  $\|a_{x_0}^k\|^{1/k} \leq \rho_{A_{x_0}}(a_{x_0}) + \varepsilon/2$ . Apply Lemma 3.1 to the element  $a^k$  of  $A$  to find a neighbourhood  $V$  of  $x_0$  such that

$$\|\chi' a^k\|^{1/k} \leq (\|a_{x_0}^k\| + (\frac{\varepsilon}{2})^k)^{1/k} \leq \|a_{x_0}^k\|^{1/k} + \frac{\varepsilon}{2}$$

for all functions  $\chi' \in \mathcal{C}_c(X)$  with  $0 \leq \chi' \leq 1$  and  $\text{supp } \chi' \subseteq V$ . Let  $\chi \in \mathcal{C}_c(X)$  such that  $0 \leq \chi \leq 1$  and  $\text{supp } \chi \subseteq V$ . Then using the above inequality for  $\chi' = \chi^k$  we obtain

$$\rho_A(\chi a) \leq \|\chi^k a^k\|^{1/k} \leq \|a_{x_0}^k\|^{1/k} + \frac{\varepsilon}{2} \leq \rho_{A_{x_0}}(a_{x_0}) + \varepsilon. \quad \square$$

**Lemma 3.3.** *Let  $\chi, \chi' \in \mathcal{C}_c(X)$  such that  $0 \leq \chi \leq \chi'$ . Let  $a \in A$ . Then  $\|\chi a\| \leq \|\chi' a\|$ .*

*Proof.* Find a function  $\chi'' \in \mathcal{C}_c(X)$  such that  $0 \leq \chi'' \leq 1$  and  $\chi'' \equiv 1$  on the support of  $\chi'$ . Let  $\varepsilon > 0$ . Then  $\chi' + \varepsilon\chi''$  satisfies  $\chi \leq \chi' + \varepsilon\chi''$ . Moreover, we have that

$$x \mapsto \frac{\chi(x)}{\chi'(x) + \varepsilon\chi''(x)}$$

defines a continuous function on the open set  $\{x' \in X : \chi''(x') > 0\}$ , and we have  $\chi(x) = 0$  for all  $x$  in the open set  $X \setminus \text{supp } \chi$ . So defining

$$\delta(x) := \begin{cases} \frac{\chi(x)}{\chi'(x) + \varepsilon\chi''(x)} & \text{if } \chi''(x) > 0, \\ 0 & \text{if } \chi(x) = 0 \end{cases}$$

for all  $x \in X$  defines a function  $\delta \in \mathcal{C}_c(X)$  such that  $0 \leq \delta \leq 1$ . We have

$$\delta(\chi' + \varepsilon\chi'')a = \chi a$$

and hence

$$\|\chi a\| = \|\delta(\chi' + \varepsilon\chi'')a\| \leq \|\delta\|_\infty \|(\chi' + \varepsilon\chi'')a\| \leq \|\chi'a\| + \varepsilon\|a\|.$$

Because this is true for all  $\varepsilon > 0$ , we can conclude that  $\|\chi a\| \leq \|\chi'a\|$ .  $\square$

**Theorem 3.4.** *For every  $a \in A$ , the function*

$$x \mapsto \rho_{A_x}(a_x)$$

*is upper semi-continuous and vanishes at infinity. If  $X$  is second countable, then*

$$\rho_A(a) = \max_{x \in X} \rho_{A_x}(a_x)$$

*for all  $a \in A$ .*

*Proof.* For the first assertion, let  $a \in A$ , let  $x_0 \in X$  and let  $\varepsilon > 0$ . Find a neighbourhood  $V$  of  $x_0$  as in Lemma 3.2. Let  $\chi \in \mathcal{C}_c(X)$  be such that  $0 \leq \chi \leq 1$ ,  $\text{supp } \chi \subseteq V$  and  $\chi \equiv 1$  on a neighbourhood  $U$  of  $x_0$ . Let  $x \in U$ . Then

$$\rho_{A_x}(a_x) \leq \rho_A(\chi a) \leq \rho_{A_{x_0}}(a_{x_0}) + \varepsilon$$

follows from  $(\chi a)_x = a_x$  and from Lemma 3.2. From this we see that  $x \mapsto \rho_{A_x}(a_x)$  is upper semi-continuous. From  $\rho_{A_x}(a_x) \leq \|a_x\|$  for all  $x \in X$  it follows that  $x \mapsto \rho_{A_x}(a_x)$  vanishes at infinity because  $x \mapsto \|a_x\|$  does.

Now for the second assertion. We first reduce to the case that  $X$  is compact. Consider the one-point compactification  $X^+$ . The  $\mathcal{C}_0(X)$ -Banach algebra  $A$  is also a  $\mathcal{C}(X^+)$ -Banach algebra with  $A_\infty = \{0\}$ . We therefore have  $\rho_{A^+}(a) = \rho_A(a)$  and  $\max_{x \in X^+} \rho_{A_x}(a_x) = \max_{x \in X} \rho_{A_x}(a_x)$  for all  $a \in A$ . If  $X$  is second countable,

then  $X^+$  is compact and metrisable. Hence it suffices to consider the case that  $X$  is compact and metrisable.

So let  $X$  be a compact metric space. Let  $a \in A$ . Define

$$m := \max_{x \in X} \rho_{A_x}(a_x).$$

We show that  $\rho_A(a) \leq m + \varepsilon$  for all  $\varepsilon > 0$ .

Let  $\varepsilon > 0$ . For all  $x \in X$  find an (open) neighbourhood  $V_x$  as in Lemma 3.2 for  $a, \varepsilon$  and  $x$ . Then  $(V_x)_{x \in X}$  is an open covering of  $X$ . Find a finite subset  $S$  of  $X$  such that  $(V_s)_{s \in S}$  is also a covering of  $X$ . Find some  $\delta > 0$  such that every subset of  $X$  of diameter less than  $\delta$  is contained in one of the sets  $V_s$ . Find a finite refinement  $\mathcal{U}$  of  $\{V_s : s \in S\}$  (i.e. a finite set of open subsets of  $X$  which covers  $X$  and such that every element of  $\mathcal{U}$  is contained in a  $V_s$ ) such that every element of  $\mathcal{U}$  has diameter less than  $\delta/2$  (to produce such a refinement first take any open cover of  $X$  by sets of diameter less than  $\delta/2$ , find a finite subcover; this finite subcover is automatically a refinement of  $\{V_s : s \in S\}$ ).

Define  $\mathcal{N} := \{\Delta \subseteq \mathcal{U} : \bigcap \Delta \neq \emptyset\}$ . This is a finite (combinatorial) simplicial complex. Note that  $\bigcup \Delta$  has diameter less than  $\delta$  for all  $\Delta \in \mathcal{N}$ , so  $\bigcup \Delta$  is contained in a  $V_s$  with  $s \in S$ .

Let  $(\chi_U)_{U \in \mathcal{U}}$  be a continuous partition of unity subordinate to the finite cover  $\mathcal{U}$ .

If  $\Delta \in \mathcal{N}$ , then  $\chi_\Delta := \sum_{U \in \Delta} \chi_U$  satisfies  $0 \leq \chi_\Delta \leq 1$  and is supported in a set  $V_s$  with  $s \in S$ , hence

$$\rho_A(\chi_\Delta a) \leq \rho_{A_s}(a_s) + \varepsilon \leq m + \varepsilon. \tag{1}$$

If  $x \in X$ , then  $\Delta_x := \{U \in \mathcal{U} : x \in U\}$  is in  $\mathcal{N}$ . So for all  $x \in X$ :

$$\sum_{U \in \mathcal{U}} \chi_U(x) = 1 = \sum_{U \in \Delta_x} \chi_U(x) = \chi_{\Delta_x}(x)$$

In particular, we have

$$1 \leq \sum_{\Delta \in \mathcal{N}} (\chi_\Delta(x))^k$$

for all  $k \in \mathbb{N}$  and  $x \in X$ . It hence follows from Lemma 3.3 that

$$\|a^k\| \leq \sum_{\Delta \in \mathcal{N}} \|(\chi_\Delta)^k a^k\|$$

for all  $k \in \mathbb{N}$ . It follows that

$$\|a^k\|^{1/k} \leq \left( \sum_{\Delta \in \mathcal{N}} \|(\chi_\Delta a)^k\| \right)^{1/k}$$

for all  $k \in \mathbb{N}$ . The left-hand side approaches  $\rho_A(a)$  if  $k \rightarrow \infty$ , the right-hand side converges to

$$\max_{\Delta \in \mathcal{N}} \rho_A(\chi_\Delta a) \stackrel{(1)}{\leq} m + \varepsilon.$$

So we have shown that

$$\rho_A(a) \leq m + \varepsilon$$

for all  $\varepsilon > 0$ , so  $\rho_A(a) \leq m$ .  $\square$

**3.2. Consequences of the spectral radius formula.** We can now prove Theorem 1.1 which we restate here for the reader's convenience (in a slightly more precise version).

**Theorem 3.5.** *Let  $A$  and  $A'$  be  $\mathcal{C}_0(X)$ -Banach algebras and let  $\varphi: A \rightarrow A'$  be a  $\mathcal{C}_0(X)$ -linear contractive homomorphism of Banach algebras. Suppose that  $\varphi_x: A_x \rightarrow A'_x$  is spectral for all  $x \in X$  and that  $X$  is second countable. Then  $\varphi$  is a spectral homomorphism. Moreover, if  $\varphi_x$  is dense for all  $x \in X$ , then  $\varphi$  is dense, too, and  $\varphi_*: K_*(A) \rightarrow K_*(A')$  is an isomorphism.*

*Proof.* If  $a \in A$ , then by Theorem 3.4,

$$\rho_{A'}(\varphi(a)) = \max_{x \in X} \rho_{A'_x}(\underbrace{\varphi(a)_x}_{=\varphi_x(a_x)}) = \max_{x \in X} \rho_{A_x}(a_x) = \rho_A(a).$$

So  $\varphi$  is spectral.

Now assume that  $\varphi(A)$  is fibrewise dense in  $A'$ .

We first consider the case that  $A'$  is locally  $\mathcal{C}_0(X)$ -convex (see the discussion in Section 2). The space  $\varphi(A)$  is not only fibrewise dense in  $A'$ , but also invariant under  $\mathcal{C}_0(X)$ , so it is dense in  $A'$  by the Stone–Weierstrass Theorem for locally  $\mathcal{C}_0(X)$ -convex  $\mathcal{C}_0(X)$ -Banach spaces (an early variant of this is Theorem 7.9 of [Hof72]; see also Proposition 2.3 in [DG83]). Hence  $\varphi_*$  is an isomorphism in this case.

Now let  $A'$  be arbitrary and let  $\iota_{A'}: A' \rightarrow \mathcal{G}(A')$  be the canonical homomorphism of  $A'$  into its Gelfand transform  $\mathcal{G}(A')$  (compare Section 2). By construction,  $\iota_{A'}$  is a fibrewise isomorphism and  $\mathcal{G}(A')$  is locally  $\mathcal{C}_0(X)$ -convex. In particular,  $\iota_{A'}$  is an isomorphism in K-theory by the preceding part of the proof. Also  $\iota_{A'} \circ \varphi$  is a fibrewise homomorphism into  $\mathcal{G}(A')$ , so it is an isomorphism in K-theory, too. So  $\varphi$  is an isomorphism in K-theory as well.  $\square$

We have shown and used the following fact in the proof of the preceding proposition, but it is certainly worth to be stated explicitly (it is analogous to a theorem by Arens, Eidlin and Novodvorskii for the Gelfand transformation of commutative Banach algebras, see [Bos90], Theorem 1.3.2, or [Tay76], Section 7.5):

**Corollary 3.6.** *Let  $X$  be second-countable. Let  $A$  be a  $\mathcal{C}_0(X)$ -Banach algebra and let  $\mathcal{G}(A)$  be the Gelfand transform of  $A$  (in the sense introduced in Section 2). Then the canonical map from  $A$  to  $\mathcal{G}(A)$  is a fibrewise isomorphism and therefore an isomorphism in K-theory:*

$$K_*(A) \cong K_*(\mathcal{G}(A)).$$

As another consequence of the spectral radius formula, we recover Proposition 2.8 of [Bla96] (at least for second countable  $X$ ):

**Corollary 3.7.** *Let  $X$  be second-countable. Let  $A$  be a  $\mathcal{C}_0(X)$ - $C^*$ -algebra. Then  $\|a\| = \max_{x \in X} \|a_x\|_{A_x}$  for all  $a \in A$ .*

*Proof.* Also the fibres of  $A$  are  $C^*$ -algebras, and we can conclude that

$$\|a\|^2 = \|a^*a\| = \rho_A(a^*a) = \max_{x \in X} \rho_{A_x}(a_x^*a_x) = \max_{x \in X} \|a_x^*a_x\|_{A_x} = \max_{x \in X} \|a_x\|_{A_x}^2. \quad \square$$

## 4. Monotone completions and their fibres

**4.1. Definition.** In [Laf02] and [Laf06], the notion of an unconditional completion was introduced which is a special case of what we propose to call a *monotone* completion. Already the article [Laf02] provides us with some interesting examples of monotone completions which are not unconditional completions.<sup>1</sup> The difference simply is that an unconditional completion is required to carry a product making it a Banach algebra whereas an unconditional completion is a Banach space without any product.

Let  $Y$  be a locally compact Hausdorff space.

**Definition 4.1.** A semi-norm  $\|\cdot\|_{\mathcal{H}}$  on  $\mathcal{C}_c(Y)$  is called *monotone* if, for all  $\varphi_1, \varphi_2 \in \mathcal{C}_c(Y)$ , the following condition holds:

$$|\varphi_1(y)| \leq |\varphi_2(y)| \implies \|\varphi_1\|_{\mathcal{H}} \leq \|\varphi_2\|_{\mathcal{H}} \quad \text{for all } y \in Y. \quad (2)$$

Let  $\mathcal{H}(Y)$  denote the (Hausdorff-)completion of  $\mathcal{C}_c(Y)$  with respect to this semi-norm; this Banach space is called a *monotone completion* of  $\mathcal{C}_c(Y)$ .

*For the rest of this section, let  $\mathcal{H}(Y)$  be a monotone completion of  $\mathcal{C}_c(Y)$ .*

For technical reasons and as for unconditional norms, we extend monotone norms to a larger class of functions on  $Y$ :

**Definition 4.2.** Let  $\mathcal{F}_c(Y)$  be the set of all (locally) bounded functions  $\varphi : Y \rightarrow \mathbb{R}$  with compact support. Let  $\mathcal{F}_c^+(Y)$  be the set of elements of  $\mathcal{F}_c(Y)$  which are non-negative. Define

$$\|\varphi\|_{\mathcal{H}} := \inf\{\|\psi\|_{\mathcal{H}} : \psi \in \mathcal{C}_c(Y), \psi \geq \varphi\}$$

for all  $\varphi \in \mathcal{F}_c^+(Y)$ .

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<sup>1</sup>For example  $H^2(G, A)$  defined after Lemme 1.6.5 or the “normalised” completions  $L_{\text{norm}}^{p,l}(G, A)$  appearing in 4.5.

Note that, by property (2), the new semi-norm agrees on  $\mathcal{C}_c^+(Y)$  with the semi-norm we started with. For all  $\varphi_1, \varphi_2, \varphi \in \mathcal{F}_c^+(Y)$  and all  $c \geq 0$ , we have

- (1)  $\varphi_1 + \varphi_2 \in \mathcal{F}_c^+(Y)$  and  $\|\varphi_1 + \varphi_2\|_{\mathcal{H}} \leq \|\varphi_1\|_{\mathcal{H}} + \|\varphi_2\|_{\mathcal{H}}$ ;
- (2)  $c\varphi \in \mathcal{F}_c^+(Y)$  and  $\|c\varphi\|_{\mathcal{H}} = c\|\varphi\|_{\mathcal{H}}$ ;
- (3) if  $\varphi_1 \leq \varphi_2$ , then  $\|\varphi_1\|_{\mathcal{H}} \leq \|\varphi_2\|_{\mathcal{H}}$ .

Hence we can use the extended semi-norm to define a semi-norm on sections of u.s.c. fields of Banach spaces.

*For the rest of this section, let  $E$  be a u.s.c. field of Banach spaces over  $Y$ .*

**Definition 4.3.** We define the following semi-norm on  $\Gamma_c(Y, E)$ :

$$\|\xi\|_{\mathcal{H}} := \|y \mapsto \|\xi(y)\|_{E_y}\|_{\mathcal{H}}.$$

The Hausdorff completion of  $\Gamma_c(Y, E)$  with respect to this semi-norm will be denoted by  $\mathcal{H}(Y, E)$ .

Note that the function  $y \mapsto \|\xi(y)\|$  appearing in the preceding definition is not necessarily continuous. However, it has compact support and is non-negative upper semi-continuous, so we can apply the extended semi-norm on  $\mathcal{F}_c^+(Y)$  to it.

If  $E$  is the trivial bundle over  $Y$  with fibre  $E_0$ , then  $\Gamma_c(Y, E)$  is  $\mathcal{C}_c(Y, E_0)$ . The completion  $\mathcal{H}(Y, E)$  of  $\mathcal{C}_c(Y, E_0)$  could hence also be denoted as  $\mathcal{H}(Y, E_0)$  and might be considered as a sort of tensor product of  $\mathcal{H}(Y)$  and  $E_0$ . If in particular  $E_0 = \mathbb{C}$ , then  $\mathcal{H}(Y, E) = \mathcal{H}(Y, \mathbb{C}) = \mathcal{H}(Y)$ .

**Definition 4.4.** Let  $F$  be another u.s.c. field of Banach spaces over  $Y$  and let  $T$  be a bounded continuous field of linear maps from  $E$  to  $F$ . Then  $\xi \mapsto T \circ \xi$  is a linear map from  $\Gamma_c(Y, E)$  to  $\Gamma_c(Y, F)$  such that  $\|T \circ \xi\|_{\mathcal{H}} \leq \|T\| \|\xi\|_{\mathcal{H}}$ . Hence  $T$  induces a canonical continuous linear map from  $\mathcal{H}(Y, E)$  to  $\mathcal{H}(Y, F)$  with norm  $\leq \|T\|$ .

This way, we define a functor from the category of u.s.c. fields of Banach spaces over  $Y$  to the category of Banach spaces, which is linear and contractive on the morphism sets.

Note that the canonical map from  $\Gamma_c(Y, E)$  to  $\mathcal{H}(Y, E)$  is continuous if we take the inductive limit topology on  $\Gamma_c(Y, E)$  and the norm topology on  $\mathcal{H}(Y, E)$ . It follows that, if a subset of  $\Gamma_c(Y, E)$  is dense in  $\Gamma_c(Y, E)$  for the inductive limit topology, then its canonical image in  $\mathcal{H}(Y, E)$  is dense for the norm topology.

**4.2. A more precise formulation of Theorem 1.2.** Let  $\pi_1: Y \times X \rightarrow Y$  and  $\pi_2: Y \times X \rightarrow X$  be the canonical projections. Note that  $\pi_1^*E = (E_y)_{(y,x) \in Y \times X}$  is an upper semi-continuous field of Banach spaces over  $Y \times X$ . The pushforward  $\pi_{1,*}(\pi_1^*E)$  is an upper semi-continuous field over  $Y$ , the fibre over  $y \in Y$  being isomorphic to  $E_y X = \mathcal{C}_0(X, E_y)$ . We denote this pushforward by  $EX = (E_y X)_{y \in Y}$ .

We form the Banach space  $\mathcal{H}(Y, EX)$  and compare it to  $\mathcal{C}_0(X, \mathcal{H}(Y, E)) = \mathcal{H}(Y, E)X$ . Note that there is a canonical contractive linear map  $\iota$  from  $\mathcal{H}(Y, EX)$  to  $\mathcal{H}(Y, E)X$ .

The second space is actually a  $\mathcal{C}_0(X)$ -Banach space. If  $x \in X$ , then the fibre of  $\mathcal{H}(Y, E)X$  over  $x$  is canonically isomorphic to  $\mathcal{H}(Y, E)$ . On the other hand, there is a canonical contractive linear map  $e_x$  from  $\mathcal{H}(Y, EX)$  to  $\mathcal{H}(Y, E)$ : If  $\text{ev}_x^E$  denotes the canonical evaluation map at  $x$  from  $EX$  to  $E$ , then  $e_x = \mathcal{H}(Y, \text{ev}_x^E)$ . The following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}(Y, EX) & \xrightarrow{\iota} & \mathcal{H}(Y, E)X \\ & \searrow e_x & \downarrow \text{ev}_x^{\mathcal{H}(Y, E)} \\ & & \mathcal{H}(Y, E). \end{array}$$

Note that there is a canonical  $\mathcal{C}_0(X)$ -structure on  $\mathcal{H}(Y, EX)$ : If  $\xi \in \Gamma_c(Y, EX)$  and  $\chi \in \mathcal{C}_0(X)$ , then  $(\chi\xi)(y) := \chi \cdot \xi(y)$  for all  $y \in Y$ ; to interpret this formula note that  $\xi(y)$  is contained in the fibre  $(EX)_y$  of  $EX$  over  $y \in Y$  which is isomorphic to  $\mathcal{C}_0(X, E_y)$  (as mentioned above), hence there is a canonical product between  $\mathcal{C}_0(X)$  and  $(EX)_y$ . The linear map  $\iota$  is clearly  $\mathcal{C}_0(X)$ -linear and hence it induces a canonical homomorphism  $\iota_x$  from the fibre  $\mathcal{H}(Y, EX)_x$  to  $(\mathcal{H}(Y, E)X)_x \cong \mathcal{H}(Y, E)$  for every  $x \in X$ . We can hence extend the above diagram to the following commutative square

$$\begin{array}{ccc} \mathcal{H}(Y, EX) & \xrightarrow{\iota} & \mathcal{H}(Y, E)X \\ \pi_x \downarrow & \searrow e_x & \downarrow \text{ev}_x^{\mathcal{H}(Y, E)} \\ \mathcal{H}(Y, EX)_x & \xrightarrow{\iota_x} & \mathcal{H}(Y, E), \end{array}$$

where we write  $\pi_x$  for the canonical projection which maps  $f \in \mathcal{H}(Y, EX)$  to  $f_x \in \mathcal{H}(Y, EX)_x$ .

So a more precise formulation of Theorem 1.2 is:

**Theorem 4.5.** *For all  $x_0 \in X$ , the map  $\iota_{x_0}: \mathcal{H}(Y, EX)_{x_0} \rightarrow \mathcal{H}(Y, E)$  is an isometric isomorphism.*

*Proof.* Let  $x_0 \in X$ . Because  $\iota(\Gamma_c(Y, EX))$  is dense in  $\mathcal{H}(Y, E)X$ , we know that  $\iota_{x_0}$  has dense image. Hence it suffices to show that  $\iota_{x_0}$  is isometric. We now show:

For all  $f \in \mathcal{H}(Y, EX)$  and all  $\varepsilon > 0$  there is a function  $\chi \in \mathcal{C}_c(X)$  such that  $0 \leq \chi \leq 1$ ,  $\chi(x_0) = 1$ , and such that

$$\|\chi f\|_{\mathcal{H}} \leq \|e_{x_0}(f)\|_{\mathcal{H}} + \varepsilon.$$

This is sufficient because  $\iota_{x_0}(f_{x_0}) = e_{x_0}(f)$  and  $f_{x_0} = (\chi f)_{x_0}$ , which implies that

$$\|\iota_x(f_{x_0})\| \leq \|f_{x_0}\| \leq \|(\chi f)_{x_0}\| = \|\chi f\|_{\mathcal{H}} \stackrel{!}{\leq} \|e_{x_0}(f)\|_{\mathcal{H}} + \varepsilon = \|\iota_{x_0}(f_{x_0})\| + \varepsilon.$$

If we prove this for arbitrary  $\varepsilon > 0$ , then we have shown that  $\iota_{x_0}$  is isometric (note that the first inequality follows from the fact that  $\|\iota_{x_0}\| \leq \|\iota\| \leq 1$ ).

We first treat the case that  $f \in \Gamma_c(Y \times X, \pi_1^*E) \subseteq \Gamma_c(Y, EX)$ . Let  $\varepsilon > 0$ . The set  $K := \pi_1(\text{supp } f) \subseteq Y$  is compact. Let  $U$  be a compact neighbourhood of  $K$  in  $Y$ . Because  $\mathcal{H}(Y)$  is a monotone completion of  $\mathcal{C}_c(Y)$ , we can find a constant  $C \geq 0$  such that  $\|\xi\|_{\mathcal{H}} \leq C \|\xi\|_{\infty}$  for all  $\xi \in \mathcal{C}_c(Y)$  such that  $\text{supp } \xi \subseteq U$ .

Because  $(y, x) \mapsto \|f(y, x) - f(y, x_0)\|_{E_y}$  is upper semi-continuous and vanishes on the compact set  $K \times \{x_0\}$ , we can choose a neighbourhood  $V$  of  $x_0$  in  $X$  such that

$$\sup_{(y,x) \in K \times V} \|f(y, x) - f(y, x_0)\|_{E_y} \leq \frac{\varepsilon}{C}.$$

Let  $\chi \in \mathcal{C}_c(X)$  be a function such that  $0 \leq \chi \leq 1$ ,  $\chi(x_0) = 1$  and  $\text{supp } \chi \subseteq V$ .

Choose a function  $\delta_K \in \mathcal{C}_c(\mathcal{G})$  such that  $0 \leq \delta_K \leq 1$ ,  $\delta_K \equiv 1$  on  $K$  and  $\text{supp } \delta_K \subseteq U$ . For all  $y \in K$  we have

$$\begin{aligned} \sup_{x \in X} \|\chi(x)f(y, x)\| &= \sup_{x \in V} \|\chi(x)f(y, x)\| \leq \sup_{x \in V} \|f(y, x)\| \\ &\leq \sup_{x \in V} (\|f(y, x) - f(y, x_0)\| + \|f(y, x_0)\|) \\ &\leq \|f(y, x_0)\| + \frac{\varepsilon}{C} \delta_K(y), \end{aligned}$$

and for  $y \in Y \setminus K$  we have

$$\sup_{x \in X} \|\chi(x)f(y, x)\| = 0 \leq \|f(y, x_0)\| + \frac{\varepsilon}{C} \delta_K(y).$$

Because  $\|\delta_K\|_{\mathcal{H}} \leq C$ , we have

$$\begin{aligned} \|\chi f\|_{\mathcal{H}} &= \|y \mapsto \sup_{x \in X} \|\chi(x)f(y, x)\|\|_{\mathcal{H}} \leq \|y \mapsto \|f(y, x_0)\| + \frac{\varepsilon}{3C} \delta_K(y)\|_{\mathcal{H}} \\ &\leq \|y \mapsto \|f(y, x_0)\|\|_{\mathcal{H}} + \frac{\varepsilon}{C} \|\delta_K\|_{\mathcal{H}} \leq \|e_{x_0}(f)\|_{\mathcal{H}} + \varepsilon. \end{aligned}$$

We now treat the general case, so let  $f$  be an arbitrary element of  $\mathcal{H}(Y, EX)$ . Let  $\varepsilon > 0$ . Then we can find an  $f' \in \Gamma_c(Y \times X, \pi_1^*E)$  such that  $\|f - f'\|_{\mathcal{H}} \leq \varepsilon/3$ . Note that this also implies  $\|e_{x_0}(f) - e_{x_0}(f')\|_{\mathcal{H}} \leq \varepsilon/3$ . By the first part of the proof we can find a function  $\chi \in \mathcal{C}_c(X)$  such that  $0 \leq \chi \leq 1$ ,  $\chi(x_0) = 1$  and

$$\|\chi f'\|_{\mathcal{H}} \leq \|e_{x_0}(f')\|_{\mathcal{H}} + \frac{\varepsilon}{3}.$$

Now

$$\|\chi f\|_{\mathcal{H}} \leq \|\chi f'\|_{\mathcal{H}} + \frac{\varepsilon}{3} \leq \|e_{x_0}(f')\|_{\mathcal{H}} + \frac{2\varepsilon}{3} \leq \|e_{x_0}(f)\|_{\mathcal{H}} + \varepsilon. \quad \square$$

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W. Paravicini, Mathematisches Institut, Universität Münster, Einsteinstr. 62,  
48149 Münster, Germany  
E-mail: W.Paravicini@uni-muenster.de