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## Isoperimetric characterization of the incenter of a triangle

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### 1 Introduction

Recently Katsuyuki Shibata introduced a new kind of center of a triangle, which he calls the *illuminating center* ([3]). It is a point that maximizes the total brightness of a triangular park  $\Omega$  obtained by a light source on that point, namely, a point that maximizes  $V_0(x) = \int_{\Omega} |x - y|^{-2} d\mu(y)$ , where  $\mu$  is the standard Lebesgue measure of  $\mathbb{R}^2$ . Unfortunately,  $V_0(x)$  is not well-defined; it diverges for any point in  $\Omega$ . In order to produce a well-defined potential, Shibata used the cut-off of the divergence of the integrand.

In [2] the author introduced the renormalization of  $\int_{\Omega} |x - y|^{\alpha-m} d\mu(y)$  (which is called the Riesz potential when  $0 < \alpha < m$ ) of a compact subset  $\Omega$  in  $\mathbb{R}^m$  which is a closure of an open set for  $\alpha \leq 0$  to obtain a one-parameter family of (*renormalized*) *potentials*  $V_{\Omega}^{(\alpha)}$ , and studied the points where the extremal values of  $V_{\Omega}^{(\alpha)}$  are attained, which we call the  $r^{\alpha-m}$ -centers of  $\Omega$ . The notion of  $r^{\alpha-m}$ -centers includes not only Shibata's illuminating center of a planar domain as an  $r^{-2}$ -center, but also the center of mass of any compact set  $\Omega \subset \mathbb{R}^m$  as  $r^2$ -center. This is because the center of mass  $x_G$  is given by

Clark Kimberling listet auf seiner Web-Seite *Encyclopedia of Triangle Centers* inzwischen weit über 5000 Dreieckszentren auf. Dort ist z.B.  $X(1)$  der Inkreismittelpunkt,  $X(2)$  der Schwerpunkt, oder  $X(54)$  der Kosnita-Punkt eines Dreiecks. Zahlreiche dieser Zentren lassen sich auf unterschiedliche Weise charakterisieren. In der vorliegenden Arbeit wird gezeigt, dass der Inkreismittelpunkt gleichzeitig eine gewisse Funktion minimiert: Dazu betrachtet man das Dreieck als Grundfläche einer Pyramide mit Spitze  $p$ . Aus deren Volumen und Oberfläche bildet man sodann einen geeigneten skaleninvarianten von  $p$  abhängigen Quotienten. Minimiert man die so definierte Funktion so fällt die Projektion des optimalen Punktes  $p$  auf die Grundfläche just in den Inkreismittelpunkt des Dreiecks.

$x_G = \int_{\Omega} y d\mu(y) / \int_{\Omega} 1 d\mu(y)$ , or equivalently by  $\int_{\Omega} (x_G - y) d\mu(y) = 0$ , which implies that it can be characterized as a unique critical point of the map  $V_{\Omega}^{(m+2)} : \mathbb{R}^m \ni x \mapsto \int_{\Omega} |x - y|^2 d\mu(y) \in \mathbb{R}$ .

Shibata announced<sup>1</sup> a theorem that an  $r^a$ -center of a non-obtuse triangle approaches the circumcenter as  $a$  goes to  $+\infty$  and to the incenter as  $a$  goes to  $-\infty$ . The proof with more generality is given in [2]. Thus, we can give interpretations of the barycenter, circumcenter, and incenter of a triangle as points that optimize a kind of potential and the limits of them.

The motivation of the theorem in this note comes from the same philosophy; to express a center as a point that optimizes a kind of potential. Our potential in this note is the ratio of the volume of the cone over a given triangle  $\Omega$  and the area of its boundary, with the former being squared and the latter cubed to make the ratio scale invariant. Then, the image of the regular projection of a vertex of a cone that optimizes this ratio is nothing but the incenter.

## 2 Cone isoperimetric center

Let  $\Omega$  be a compact set which is a closure of an open subset of  $\mathbb{R}^2$  with a piecewise  $C^1$  boundary  $\partial\Omega$ . We assume that  $\mathbb{R}^2$  is embedded in  $\mathbb{R}^3$  in a standard way;  $\mathbb{R}^2 = \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_i \in \mathbb{R}\}$ . Let  $\Pi_h$  denote a level plane in  $\mathbb{R}^3$  with height  $h > 0$ ,  $\Pi_h = \{x_3 = h\}$ , and  $C_p$  a cone over  $\Omega$  with vertex  $p \in \Pi_h$ ,  $C_p = \{tx + (1-t)p \mid x \in \Omega, 0 \leq t \leq 1\}$ . Let  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the regular projection.

### Definition 2.1.

- (1) Let  $p_h$  be a point in  $\Pi_h$  where the minimum value of a function  $\Pi_h \ni p \mapsto \text{Area}(\partial C_p)$  is attained. We call  $\pi(p_h)$  a *cone isoperimetric center of  $\Omega$  of height  $h$* .
- (2) Let  $p$  be a point in  $\mathbb{R}_+^3 = \{x_3 > 0\}$  that gives the minimum value of a function

$$f(p) = \frac{(\text{Area}(\partial C_p))^3}{(\text{Vol}(C_p))^2}.$$

We call  $C_p$  an *isoperimetrically optimal cone* and  $\pi(p)$  a *cone isoperimetric center of  $\Omega$* .

**Lemma 2.2.** *Let  $\triangle ABC$  be a triangle. Then there exists a cone isoperimetric center of height  $h$  for any  $h > 0$ .*

*Proof.* Let  $S$  be the area, and  $a$ ,  $b$ , and  $c$  the lengths of the edges  $BC$ ,  $CA$ , and  $AB$ , respectively. Fix  $h > 0$ . Let  $P \in \Pi_h$  be a point and  $D = \pi(P)$ . Let  $u$ ,  $v$ , and  $w$  be the distances with signs between  $D$  and the lines  $\overline{BC}$ ,  $\overline{CA}$ , and  $\overline{AB}$ , respectively. The signs of  $u$ ,  $v$ , and  $w$  are given as follows. We put  $u > 0$  if  $D$  and  $A$  are in the same half-plane cut out by the line  $\overline{BC}$ . Remark that the position of  $D$  is determined uniquely by  $u$  and  $v$ .

1. at 2010 Autumn Meetings of the Mathematical Society of Japan

Then the area of the triangle  $\triangle ABC$  is given by  $S = \frac{1}{2}(au + bv + cw)$ , and the area of the boundary of the cone is given by

$$\text{Area}(\partial C_P) = S + \frac{1}{2} \left( a\sqrt{u^2 + h^2} + b\sqrt{v^2 + h^2} + c\sqrt{w^2 + h^2} \right). \quad (1)$$

Let the right-hand side of (1) be denoted by  $\psi(D)$ . Then, it takes the value  $S + \frac{1}{2}(a + b + c)\sqrt{r^2 + h^2}$  at the incenter  $I$ , where  $r$  is the radius of the inscribed circle. Put

$$\rho = \frac{a + b + c}{\min\{a, b, c\}} \sqrt{r^2 + h^2}.$$

Let  $\bar{N}_\rho(\overline{BC})$  be the set of points so that the distance to the line  $\overline{BC}$  is not greater than  $\rho$ , namely, a closed strip with central axis  $\overline{BC}$  which is  $2\rho$  wide. Two other strips,  $\bar{N}_\rho(\overline{CA})$  and  $\bar{N}_\rho(\overline{AB})$ , can be defined similarly. Put  $K = \bar{N}_\rho(\overline{BC}) \cap \bar{N}_\rho(\overline{CA}) \cap \bar{N}_\rho(\overline{AB})$ . Then  $K$  is a compact set containing  $I$ .

Suppose  $D \notin K$ . Then at least one of  $|u|$ ,  $|v|$ , and  $|w|$  is greater than  $\rho$ . Therefore,

$$\psi(D) > S + \frac{1}{2} \min\{a, b, c\} \sqrt{\rho^2 + h^2} > S + \frac{1}{2} \min\{a, b, c\} \rho = \psi(I),$$

which implies  $\inf_{D' \in K} \psi(D') = \inf_{D'' \in \mathbb{R}^2} \psi(D'')$ . Since  $\psi$  is continuous and  $K$  is compact, there is a point  $D \in K$  where  $\inf_{D' \in K} \psi(D')$  is attained.

It follows that  $\inf_{D'' \in \mathbb{R}^2} \psi(D'')$  is also attained at  $D$ .  $\square$

**Theorem 2.3.** *Let  $\triangle ABC$  be a triangle. The cone isoperimetric center of height  $h$  coincides with the incenter for any  $h > 0$ . The height of the isoperimetrically optimal cone is  $2\sqrt{2}$  times the radius of the inscribed circle.*

*Proof.* (1) Let us use the same notation as in Lemma 2.2.

Let  $D_h$  be a cone isoperimetric center of  $\triangle ABC$  of height  $h$ , and  $u_h$ ,  $v_h$ , and  $w_h$  be the signed distances between  $D_h$  and the lines  $\overline{BC}$ ,  $\overline{CA}$ , and  $\overline{AB}$ , respectively. Then the pair  $(u_h, v_h)$  minimizes a function

$$F(u, v) = a\sqrt{u^2 + h^2} + b\sqrt{v^2 + h^2} + c\sqrt{\left(\frac{2S - au - bv}{c}\right)^2 + h^2}.$$

Therefore, when  $(u, v, w) = (u_h, v_h, w_h)$  we have

$$\begin{aligned} F_u(u, v) &= \frac{au}{\sqrt{u^2 + h^2}} + \frac{cw}{\sqrt{w^2 + h^2}} \cdot \left(-\frac{a}{c}\right) = 0, \\ F_v(u, v) &= \frac{bv}{\sqrt{v^2 + h^2}} + \frac{cw}{\sqrt{w^2 + h^2}} \cdot \left(-\frac{b}{c}\right) = 0, \end{aligned}$$

which implies

$$\frac{u}{\sqrt{u^2 + h^2}} = \frac{v}{\sqrt{v^2 + h^2}} = \frac{w}{\sqrt{w^2 + h^2}}. \quad (2)$$

Remark that the above holds only when  $u$ ,  $v$ , and  $w$  are all positive, implying that  $D_h$  is in the interior of  $\triangle ABC$ . The equation (2) means that three angles between the  $xy$ -plane and three planes through  $PAB$ ,  $PBC$ , and  $PCA$  are all equal. Therefore, each pair of the three planes is symmetric in a plane which is orthogonal to the  $xy$ -plane and contains the intersection line of the pair. These three symmetries show that the three lines  $D_hA$ ,  $D_hB$ , and  $D_hC$ , which are the intersections of the  $xy$ -plane and the three planes of the symmetries, are the angle bisectors of  $\angle A$ ,  $\angle B$ , and  $\angle C$ , respectively. It follows that  $D_h$  coincides with the incenter of  $\triangle ABC$ .

(2) The second statement follows from elementary calculus. Let  $r$  be the radius of the inscribed circle. Put  $P_h = \pi^{-1}(D_h) \cap \Pi_h$ , then

$$\text{Area}(\partial C_{P_h}) = S + \frac{1}{2}(a+b+c)r\sqrt{1+\left(\frac{h}{r}\right)^2} = S\left(1+\sqrt{1+\left(\frac{h}{r}\right)^2}\right).$$

As  $\text{Vol}(C_{P_h}) = \frac{1}{3}Sh$ ,

$$f(P_h) = \frac{(\text{Area}(\partial C_{P_h}))^3}{(\text{Vol}(C_{P_h}))^2} = 9S \frac{\left(1+\sqrt{1+\left(\frac{h}{r}\right)^2}\right)^3}{h^2} = \frac{9S}{r^2} \cdot \frac{\left(1+\sqrt{1+\left(\frac{h}{r}\right)^2}\right)^3}{\left(\frac{h}{r}\right)^2}.$$

Since  $\varphi(t) = \frac{(1+\sqrt{1+t^2})^3}{t^2}$  ( $t > 0$ ) takes the minimum at  $t = 2\sqrt{2}$ , it completes the proof.  $\square$

**Remark 2.4.** The above theorem means that the cone isoperimetric center of height  $h$  is identically the same for any  $h > 0$  and that it coincides with the limit of  $r^a$ -center as  $a$  goes to  $-\infty$  for triangles. But it does not hold in general as an example below shows.

Let us call a point an *asymptotic  $r^{-\infty}$ -center* of  $\Omega$  if it is the limit of a convergent sequence of  $r^{a_i}$ -centers with  $a_i \rightarrow -\infty$  as  $i \rightarrow +\infty$ . We showed in [2] that an asymptotic  $r^{-\infty}$ -center is a *max-min point* of  $\Omega$ , by which we mean a point that gives the supremum of a map  $\mathbb{R}^2 \ni x \mapsto \min_{y \in \overline{\Omega^c}} |y-x| \in \mathbb{R}$ , where  $\overline{\Omega^c}$  denotes the closure of the complement of  $\Omega$ . We remark that an  $r^a$ -center ( $a \leq -2$ ) and a max-min point are not necessarily unique. To see this, it is enough to consider a disjoint union of two rectangles, say,  $\Omega' = \{(\xi, \eta) \mid 1 \leq |\xi| \leq 2, |\eta| \leq 2\}$ .

Let  $\Omega$  be a trapezoid given by  $\Omega = \{(\xi, \eta) \mid 0 \leq \xi \leq 2, |\eta| \leq 1 + \frac{1}{2}\xi\}$ . It is easy to see that a cone isoperimetric center of height  $h$  is on the  $\xi$ -axis for any  $h$ . Let it be given by  $(\xi_h, 0)$ . Numerical experiments show that  $\xi_1 \sim 0.9169$ ,  $\xi_2 \sim 0.9079$ ,  $\xi_3 \sim 0.9045$ , and  $\xi_4 \sim 0.9031$ , and the minimum of the ratio  $f$  is attained at  $h \sim 3.250$  when  $\xi_h \sim 0.90405$ . On the other hand, an asymptotic  $r^{-\infty}$ -center is  $(1, 0)$ . This is because the set of max-min points is  $\{(1, \eta) \mid |\eta| \leq \frac{3}{2} - \frac{\sqrt{3}}{2}\}$  whereas any  $r^a$ -center is contained in  $\{(\xi, 0) \mid 1 \leq \xi \leq \frac{7}{4}\}$  for any  $a$  by the symmetry argument (based on the moving plane method [1]) explained in [2], and the point  $(1, 0)$  is the unique intersection point of these sets.

**References**

- [1] Gidas, B.; Ni, W.M.; Nirenberg, L.: Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* 68 (1979), 209–243.
- [2] O'Hara, J.: Renormalization of potentials and generalized centers. To appear in *Adv. Appl. Math.*
- [3] Shibata, K.: *Where should a streetlight be placed in a triangle-shaped park? Elementary integro-differential geometric optics.*  
Available at <http://www1.rsp.fukuoka-u.ac.jp/kototoi/shibataaleph-sjs.pdf>

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