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The Feuerbach circle and the other remarkable circles of the cyclic polygons

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1 Introduction

It is very well known that a triangle ABC has its Feuerbach circle (nine point circle). The Feuerbach circle is a circle that passes through the midpoints E_A, E_B, and E_C of the segments that join the vertices and the orthocenter H. These points are commonly referred to as the Euler points; we will refer the center of the Feuerbach circle as F. Also, the following properties of the Feuerbach circle are very well-known; see Fig. 1:

> The orthocenter H, triangle centroid G, circumcenter O, and the Feuerbach center F are aligned (Euler line). In fact 243 Kimberling centers are lying on the Euler line.
>
> The cross ratio $(O, F, G, H) = -1$; that is: O, F, G, and H form a harmonic range with $OG = \frac{1}{2}GH$, $OG = \frac{1}{3}OH$, $OF = \frac{1}{2}OH$, $FG = \frac{1}{6}HO$. The simple ratio $(O, H, F) = \frac{OF}{HF} = -1$.

Der Feuerbachkreis eines Dreiecks geht bekanntlich durch die Mittelpunkte der Seiten, die Fusspunkte der Höhen sowie durch die Eulerpunkte des Dreiecks. Darüber hinaus liegen der Mittelpunkt des Feuerbachkreises, der Höhenschnittpunkt und der Schwerpunkt des Dreiecks auf einer Geraden, der sogenannten Eulergeraden. In dem nachfolgenden Beitrag verallgemeinert der Autor diesen Sachverhalt auf zyklische N-Ecke ($N \geq 3$), d.h. konvexe N-Gone, die einen Umkreis besitzen. Dementsprechend findet er für das zur Diskussion stehende zyklische N-Eck unter anderem ein Analogon des Feuerbachkreises, dessen Zentrum zusammen mit dem Umkreiszentrum und dem Schwerpunkt des N-Ecks auf einer Geraden liegt.

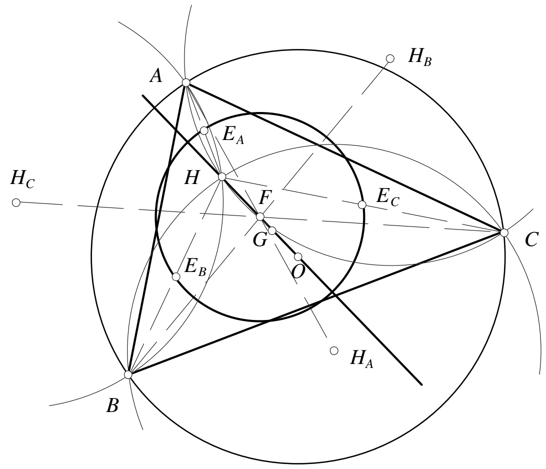

Fig. 1

(1*) The intersection of the three lines AH_A, BH_B, and CH_C is F; where H_A, H_B, and H_C are orthocenters of the segments BC, AC, and AB, respectively. These orthocenters are symmetric to O with respect to the sides BC, AC, and AB, respectively (see [3] for why these points are called orthocenters).

(2*) The radius of the Feuerbach circle is one-half the circumradius r.

For the above statements see classical works as [4], [5], and books of geometry as [8]. For much more properties of the Feuerbach circle see the $X(5)$ center in C. Kimberling's Encyclopedia of triangle centers [6] and [7].

In this paper we will refer the set of sentences (1*), (2*) as Definition 1, "Def. 1".

We refer the circle $\odot(F, \frac{r}{2})$, center F and radius $\frac{r}{2}$, where F and r are defined in "Def. 1", as the Feuerbach circle. This definition is totally different from the definition of the nine point circle given by S.N. Collings [1], [2]; and the difference resides in the different definition of the orthocenter point H of the polygon.

In this work we want to show that the Feuerbach circle is not a special entity of the triangles, but it is a general entity of a cyclic polygon with n sides, n-polygon with $n \geq 3$. But, in fact, a cyclic polygon has an infinite quantity of remarkable circles such that all their centers are aligned (Euler line).

2 Feuerbach circle

In this section we want to show an easy way to generalize the results of the introduction for a cyclic n-polygon, and we summarize the results in form of a theorem at the end of the section.

2.1 Preliminaries

1. Evidently an n-polygon has circumcenter O if it is a cyclic n-polygon, i.e., if it is a polygon with vertices upon which a circle can be circumscribed. The center of the circumcircle is O. We will refer the circumradius as r.
2. If we use Cartesian coordinates and such that $r = 1$, the vertices of the cyclic n-polygon $A_1A_2 \ldots A_n$ are $A_i = (\cos\alpha_i, \sin\alpha_i)$ then we have trivially that $O = (0, 0)$ and the polygon centroid is $G = \left(\frac{1}{n}\sum_{i=1}^{n}\cos\alpha_i, \frac{1}{n}\sum_{i=1}^{n}\sin\alpha_i\right)$.
3. If the polygon is a triangle $A_1A_2A_3$ we can see, for example in [3], that the triangle orthocenter H is the intersection of the circles $\bigcap_{j=1}^{3}\odot(H_j, r)$. That is to say: the circles of radius r and center the orthocenter, H_j, of the side $A_iA_k = A_i\widehat{A_j}A_k = \{A_1, A_2, A_3\} \setminus \{A_j\}$ opposite to vertex A_j are concurrent in a point, and this point is the orthocenter H of $A_1A_2A_3$, see Fig. 1. If $A_j = (\cos\alpha_j, \sin\alpha_j)$ we have
$$H_j = \left(-\cos\alpha_j + \sum\nolimits_{i=1}^{3}\cos\alpha_i, -\sin\alpha_j + \sum\nolimits_{i=1}^{3}\sin\alpha_i\right),$$
and
$$H = \left(\sum\nolimits_{i=1}^{3}\cos\alpha_i, \sum\nolimits_{i=1}^{3}\sin\alpha_i\right).$$
4. With the same notations of above we have that for $A_1A_2A_3$ the point
$$F = \left(\frac{1}{2}\sum\nolimits_{i=1}^{3}\cos\alpha_i, \frac{1}{2}\sum\nolimits_{i=1}^{3}\sin\alpha_i\right)$$
is its triangle Feuerbach center.

2.2 Considerations

Now we consider the above propriety number 3 and define the orthocenter by induction as the intersection of circles; that is: for a cyclic n-polygon $A_1A_2 \ldots A_n$ with $r = 1$, $n \geq 3$, we consider the j-set of $n-1$ points $A_1A_2 \ldots \widehat{A_j} \ldots A_n = \{A_1, A_2, \ldots, A_n\} \setminus \{A_j\}$. For this j-set we have, by induction, its orthocenter point $H_j = (-\cos\alpha_j + \sum_{i=1}^{n}\cos\alpha_i, -\sin\alpha_j + \sum_{i=1}^{n}\sin\alpha_i)$.

It is easy to see that the point $\left(\sum_{i=1}^{n}\cos\alpha_i, \sum_{i=1}^{n}\sin\alpha_i\right) \in \odot(H_j, 1)$; we refer this point as H. Then $H = \bigcap_{j=1}^{n}\odot(H_j, 1)$, and this intersection in this paper will be the Definition 2, "Def. 2". From now, we will say that the cyclic n-polygon has an orthocenter point H defined as "Def. 2", see Fig. 2.

Then, with the above expressions of O, G, and H, it is trivial to see that the cyclic n-polygon has an Euler line because O, G, and H are aligned; and we have the simple ratio $(O, G, H) = \frac{n}{n-1}$, see Fig. 2. The existence of this line was proved for cyclic quadrilaterals, and also was cited without proof with this simple ratio for n-polygons, by M. Dalcín in [3].

The line A_jH_j has equation $(\cos\alpha_j, \sin\alpha_j) + \lambda(-2\cos\alpha_j + \sum_{i=1}^{n}\cos\alpha_i, -2\sin\alpha_j + \sum_{i=1}^{n}\sin\alpha_i)$; then, with $\lambda = \frac{1}{2}$, it is easy to see that $\bigcap_{j=1}^{n} A_jH_j$ exists and $\bigcap_{j=1}^{n} A_jH_j =$

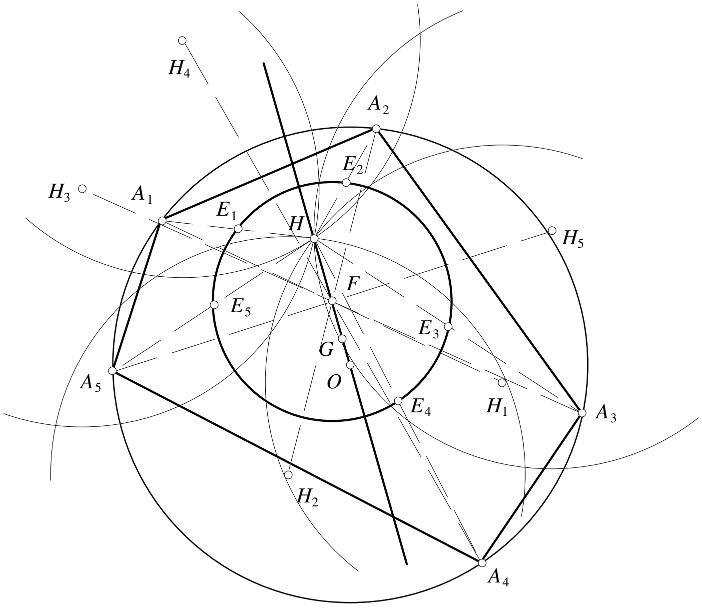

Fig. 2

$\left(\frac{1}{2}\sum_{i=1}^{n}\cos\alpha_i, \frac{1}{2}\sum_{i=1}^{n}\sin\alpha_i\right)$, we will refer this point as F, see Fig. 2. Therefore, we have that F is on the Euler line and the cross ratio $(O, F, G, H) = \frac{(O,F,G)}{(O,F,H)} = \frac{OG/FG}{OH/FH} = \frac{-\left(\frac{1}{n}/\frac{n-2}{2n}\right)}{1/\frac{1}{2}} = -\frac{1}{n-2}$.

The midpoint E_j of the segment A_jH (Euler point) is $\frac{1}{2}(\cos\alpha_j + \sum_{i=1}^{n}\cos\alpha_i, \sin\alpha_j + \sum_{i=1}^{n}\sin\alpha_i)$. Trivially we can test that $E_j \in \odot\left(F, \frac{1}{2}\right)$ because the equation of this circle is $\frac{1}{4} = \left(x - \frac{1}{2}\sum_{i=1}^{n}\cos\alpha_i\right)^2 + \left(y - \frac{1}{2}\sum_{i=1}^{n}\sin\alpha_i\right)^2$. Therefore this circle $\odot\left(F, \frac{1}{2}\right)$, see Fig. 2, is the Feuerbach circle of the cyclic n-polygon and its radius is one-half the circumradius. This circle definition in this paper will be the Definition 3, "Def. 3".

Then, we can summarize the results in the following:

Theorem 1 *Let $A_1A_2\ldots A_n$ be a cyclic n-polygon, $n \geq 3$, with circumradius r. Then:*

a) *It has an orthocenter point $H = \bigcap_{j=1}^{n}\odot(H_j, r)$, where H_j is the orthocenter of $A_1A_2\ldots\widehat{A_j}\ldots A_n = \{A_1, A_2, \ldots, A_n\} \setminus \{A_j\}$.*

b) *Its orthocenter H, its centroid G, and its circumcenter O, are aligned; therefore it has an Euler line. The simple ratio $(O, G, H) = \frac{n}{n-1}$.*

c) *Its midpoints E_j of the segments that join the vertices and the orthocenter H, the Euler points, are concyclic; therefore it has a Feuerbach circle $\odot(F, \rho)$.*

d) *The radius ρ of its Feuerbach circle is one-half the circumradius, $\rho = \frac{r}{2}$.*

e) *The center F of its Feuerbach circle is on the Euler line, and the cross ratio* $(O, F, G, H) = -\frac{1}{n-2}$.

f) *The intersection of the lines* $\bigcap_{j=1}^{n} A_j H_j = F$.

3 Difference and relations with the Collings nine point circle

3.1 Difference

In 1967, S.N. Collings [1], [2] defined the orthocenter in a different way of "Def. 2".

Collings' definition for the orthocenter H^c is: First $H^c = H$ with $n = 3$. Now, for a cyclic n-polygon $A_1A_2 \dots A_n$, $n \geq 4$, we consider the j-set of $n-1$ points $A_1A_2 \dots \widehat{A_j} \dots A_n$. For this j-set we have, by induction, its orthocenter point H^j, then Collings proves, by induction, that the lines A_jH^j concur in a point that he refers as orthocenter point $H^c = \bigcap_{j=1}^{n} A_jH^j$ of the n-polygon. This definition is different from the definition "Def. 2" and this orthocenter point H^c is different from the orthocenter point H for $n \geq 4$.

Collings proves that O, G, and H^c are aligned. Then trivially, with part b) of Theorem 1: O, G, H, and H^c are aligned (Euler line for cyclic n-polygon). He proves that the simple ratio $(O, H^c, G) = \frac{OG}{H^cG} = -\frac{n-2}{2}$.

Collings' definition for the nine point circle ν is: First $\nu = \odot(F, \frac{r}{2})$ with $n = 3$. Now, for a cyclic n-polygon $A_1\ A_2 \dots A_n$, $n \geq 4$, we consider the j-set of $n-1$ points $A_1A_2 \dots \widehat{A_j} \dots A_n$. For this j-set we have its centroid point G_j, then Collings proves that all centroids G_j lie on a circle that he refers as nine point circle ν of the n-polygon. Collings names this circle ν: nine point circle of the n-polygon. This definition is different from the definition "Def. 3" and this nine point circle $\nu = \odot(N, s)$ (N and s are the center and the radius of ν) is different from the Feuerbach circle $\odot(F, \frac{r}{2})$ for $n \geq 4$.

Collings proves that O, G, and N are aligned. Then trivially, with part e) of Theorem 1: O, G, H, H^c, F, and N are aligned. He proves that $s = \frac{r}{n-1}$, and the simple ratio $(O, N, G) = \frac{OG}{NG} = -\frac{n-1}{1}$. Therefore, the cross ratio

$$(O, N, G, H^c) = 1 - (O, G, N, H^c) = 1 - \frac{(O, G, N)}{(O, G, H^c)} = 1 - \frac{1 - (O, N, G)}{1 - (O, H^c, G)} = -1.$$

We will refer: circle $\odot(N, \frac{r}{n-1})$, points H^j, point H^c, and center N, as: circle, points, orthocenter and center, of Collings, respectively; see Fig. 3.

We will refer the circle $\odot(F, \frac{r}{2})$ as Feuerbach circle because, for us, the most important property for the Feuerbach circle of a triangle (nine point circle of a triangle) is that the radius of the Feuerbach circle is one-half the circumradius, see part d) of Theorem 1.

3.2 Relations

For $n \geq 4$, we consider the j-set of $n-1$ points $A_1A_2 \dots \widehat{A_j} \dots A_n$ of the cyclic n-polygon, $r = 1$. For this j-set we have the point $\overline{H}^j = \frac{1}{n-3}\left(-\cos\alpha_j + \sum_{i=1}^{n} \cos\alpha_i,\right.$

$- \sin\alpha_j + \sum_{i=1}^{n} \sin\alpha_i\big)$ with $\overline{H}^j = H_j$ if $n = 4$. The line $A_j\overline{H}^j$ has equation

$$(\cos\alpha_j, \sin\alpha_j) + \frac{\lambda}{n-3}\left((2-n)\cos\alpha_j + \sum\nolimits_{i=1}^{n} \cos\alpha_i,\, (2-n)\sin\alpha_j + \sum\nolimits_{i=1}^{n} \sin\alpha_i\right);$$

then, with $\lambda = \frac{n-3}{n-2}$, it is easy to see that $\bigcap_{j=1}^{n} A_j\overline{H}^j$ exists, and

$$\bigcap\nolimits_{j=1}^{n} A_j\overline{H}^j = \frac{1}{n-2}\left(\sum\nolimits_{i=1}^{n} \cos\alpha_i,\ \sum\nolimits_{i=1}^{n} \sin\alpha_i\right).$$

Therefore $\overline{H}^j = H^j$ and $\bigcap_{j=1}^{n} A_jH^j = H^c$.

We can consider the point $C = \frac{1}{n-3}\left(\sum_{i=1}^{n} \cos\alpha_i, \sum_{i=1}^{n} \sin\alpha_i\right)$. Then now it is trivial that the Collings points H^j are concyclic and $H^j \in \odot(C, \frac{1}{n-3})$, see Fig. 3. The point C is on the Euler line; $C = H$ if $n = 4$, and $C = F$ if $n = 5$ see Fig. 3.

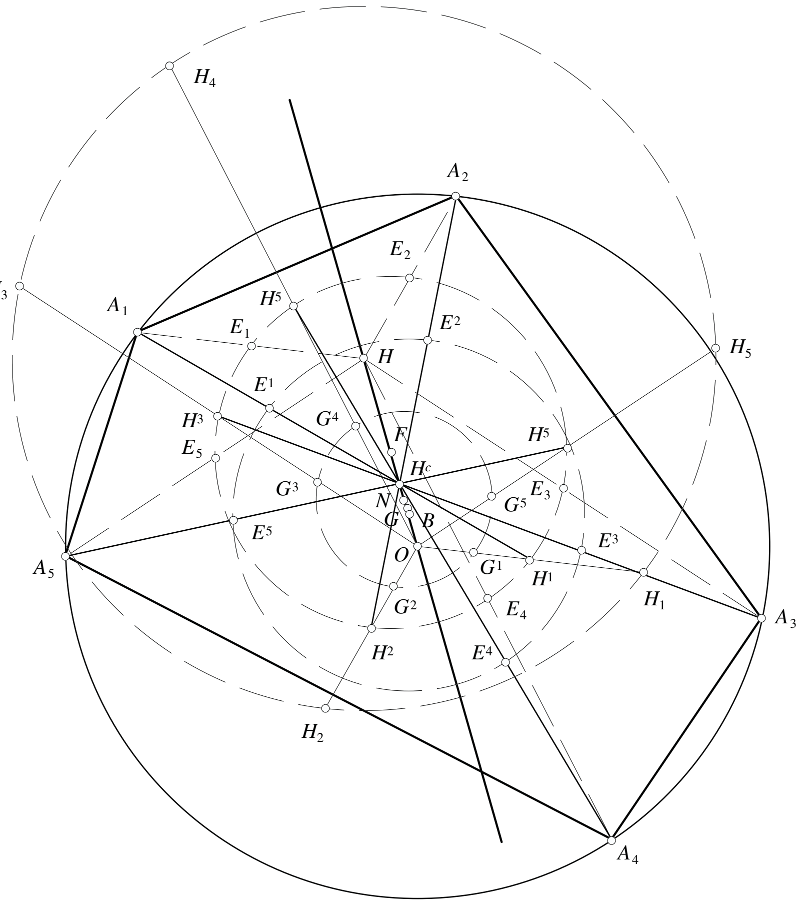

Fig. 3

Clearly, $G_j = \frac{1}{n-1}\left(-\cos\alpha_j + \sum_{i=1}^{n}\cos\alpha_i, -\sin\alpha_j + \sum_{i=1}^{n}\sin\alpha_i\right)$ and all G_j are on the circle with center $\frac{1}{n-1}\left(\sum_{i=1}^{n}\cos\alpha_i, \sum_{i=1}^{n}\sin\alpha_i\right)$ and radius $\frac{1}{n-1}$; then, this center is N.

It is trivial to see that the points O, G_j, H_j, and H^j are aligned.

With $n = 5$ then $H^j = \frac{1}{2}\left(-\cos\alpha_j + \sum_{i=1}^{n}\cos\alpha_i, -\sin\alpha_j + \sum_{i=1}^{n}\sin\alpha_i\right)$ and $E_j = \frac{1}{2}\left(\cos\alpha_j + \sum_{i=1}^{n}\cos\alpha_i, \sin\alpha_j + \sum_{i=1}^{n}\sin\alpha_i\right)$ are concyclic on the Feuerbach circle; see Fig. 3.

We can consider the Euler-Collings points; that is to say, the midpoints E^j of the segment A_jH^c, and they are the points $E^j = \frac{1}{2}\left(\cos\alpha_j + \frac{1}{n-2}\sum_{i=1}^{n}\cos\alpha_i, \sin\alpha_j + \frac{1}{n-2}\sum_{i=1}^{n}\sin\alpha_i\right)$. Trivially we can test that $E^j \in \odot\left(B, \frac{1}{2}\right)$ where B is the mid point of the segment OH^c, that is $B = \frac{1}{2n-4}\left(\sum_{i=1}^{n}\cos\alpha_i, \sum_{i=1}^{n}\sin\alpha_i\right)$. Therefore we refer $\odot\left(B, \frac{1}{2}\right)$ as Feuerbach-Collings circle of the cyclic n-polygon because its radius is one-half the circumradius; see Figs. 3, 4.

Then, we can summarize the relations in the following theorem; parts a), b) and c) are in [1], the other parts are new:

Theorem 2 *Let $A_1A_2\dots A_n$ be a cyclic n-polygon, $n \geq 4$, with circumradius r.*

Then:

a) *It has a Collings orthocenter point $H^c = \bigcap_{j=1}^{n} A_jH^j$, where H^j is, by induction, the Collings orthocenter point of $A_1A_2\dots\widehat{A_j}\dots A_n$. If $n = 3$, then $H^c = H$.*

b) *The centroid points G_j of $A_1A_2\dots\widehat{A_j}\dots A_n$ are concyclic; therefore it has a nine point circle $\odot(N, \rho)$, that we refer as Collings circle. If $n = 3$, then $\odot(N, \rho) = \odot(F, \frac{r}{2})$.*

c) $\rho = \frac{r}{n-1}$, $(O, N, G, H^c) = -1$.

e) *H, G, O, N, and H^c are aligned.*

f) *Its Collings points H^j are concyclic on the circle $\odot(C, \frac{r}{n-3})$. And the point $C = O + \frac{1}{n-3}\overrightarrow{OH}$.*

g) *If $n = 5$ then $C = F$ and the points $\{H^j, E_j\}_{j=1}^{n}$ are concyclic.*

h) *Its Euler-Collings points, that is to say, the midpoints E^j of the segments A_jH^c, are concyclic on the circle $\odot(B, \frac{1}{2})$. The point $B = O + \frac{1}{2n-4}\overrightarrow{OH}$ (i.e., B is the mid point of segment OH^c), and $E^j = O + \frac{1}{2}(\overrightarrow{OH^c} + \overrightarrow{OA_j})$.*

4 Infinite quantity of remarkable circles

We can rewrite, without coordinates, all the points that we have consider in the above sections and we have:

For a cyclic n-polygon $A_1A_2\dots A_n$ with $n \geq 4$ and circumradius r:

$$
\begin{array}{ll}
H = O + \overrightarrow{OH} & H_j = O + (\overrightarrow{OH} - \overrightarrow{OA_j}) \\
F = O + \frac{1}{2}\overrightarrow{OH} & E_j = O + \frac{1}{2}(\overrightarrow{OH} + \overrightarrow{OA_j}) \\
C = O + \frac{1}{n-3}\overrightarrow{OH} & H^j = O + \frac{1}{n-3}(\overrightarrow{OH} - \overrightarrow{OA_j}) \\
H^c = O + \frac{1}{n-2}\overrightarrow{OH} & \\
N = O + \frac{1}{n-1}\overrightarrow{OH} & G_j = O + \frac{1}{n-1}(\overrightarrow{OH} - \overrightarrow{OA_j}) \\
G = O + \frac{1}{n}\overrightarrow{OH} & \\
B = O + \frac{1}{2}\overrightarrow{OH^c} = O + \frac{1}{2n-4}\overrightarrow{OH} & E^j = O + \frac{1}{2}(\overrightarrow{OH^c} + \overrightarrow{OA_j})
\end{array}
$$

And it is enough to see the above table to easily check that:

1. The orthocenter points H_j are concyclic in the circle $\odot(H, r)$.
2. The Euler points E_j are concyclic in the Feuerbach circle $\odot(F, \frac{r}{2})$.
3. The Collings points H^j are concyclic in the circle $\odot(C, \frac{r}{n-3})$.
4. The centroid points G_j are concyclic in the circle $\odot(N, \frac{r}{n-1})$.
5. The Euler-Collings points E^j are concyclic in the Feuerbach-Collings circle $\odot(B, \frac{r}{2})$.
6. If $n = 4$ then $H = C$, $F = H^c$, the orthocenter points H_j and Collings points H^j are the same.
7. If $n = 5$ then $F = C$, the Euler points E_j and the Collings points H^j are concyclic in the Feuerbach circle $\odot(F, \frac{r}{2})$.
8. The points O, H, F, C, H^c, N, G, B are aligned (Euler line).
9. The points O, H_j, H^j, G_j are aligned.

But in general we can see easily that the cyclic n-polygon has a infinite quantity of remarkable circles, centers and points.

We can write the following

Theorem 3 *Let $A_1A_2\dots A_n$ be a cyclic n-polygon, $n \geq 4$, with circumradius r. Then:*

a) *The points O, H, F, C, H^c, N, G, B are aligned (Euler line).*
b) *The points O, H_j, H^j, G_j are aligned.*
c) *The infinite quantity of remarkable centers $\{\gamma_k = O + \frac{1}{k}\overrightarrow{OH} \mid k \in \mathbb{N}\}$ are aligned.*
d) *The remarkable set of points $\Pi_k^- = \{\pi_{kj}^- = O + \frac{1}{k}(\overrightarrow{OH} - \overrightarrow{OA_j}) \mid 1 \leq j \leq n\}$ lies on the remarkable circle $\odot(\gamma_k, \frac{r}{k})$, for each $k \in \mathbb{N}$.*

e) *The remarkable set of points* $\Pi_k^+ = \{\pi_{kj}^+ = O + \frac{1}{k}(\overrightarrow{OH} + \overrightarrow{OA_j}) \mid 1 \le j \le n\}$ *lies on the remarkable circle* $\odot(\gamma_k, \frac{r}{k})$*, for each* $k \in \mathbb{N}$.

f) *Then, the two sets of points* Π_k^-, Π_k^+ *are concyclic, for all* $k \in \mathbb{N}$*, on the circle* $\odot(\gamma_k, \frac{r}{k})$.

g) *The remarkable set of points* $\Pi_k^{c+} = \{\pi_{kj}^{c+} = O + \frac{1}{k}(\overrightarrow{OH^c} + \overrightarrow{OA_j}) \mid 1 \le j \le n\}$ *lies on the remarkable circle* $\odot(\gamma_k^c, \frac{r}{k})$*, for all* $k \in \mathbb{N}$. *The center* $\gamma_k^c = O + \frac{1}{k}\overrightarrow{OH^c} = O + \frac{1}{k(n-2)}\overrightarrow{OH} = \gamma_{k(n-2)}$.

h) *The points* O, π_{kj}^+*, for all* $k \in \mathbb{N}$*, are aligned with fixed* j.

i) *The points* O, π_{kj}^-*, for all* $k \in \mathbb{N}$*, are aligned with fixed* j.

j) *The points* O, π_{kj}^{c+}*, for all* $k \in \mathbb{N}$*, are aligned with fixed* j.

k) *And in a more general situation: The remarkable set of points* $\Pi_k^{m\pm} = \{\pi_{kj}^{m\pm} = O + \frac{1}{k}(\frac{1}{m}\overrightarrow{OH} \pm \overrightarrow{OA_j}) \mid 1 \le j \le n\}$ *lies on the remarkable circle* $\odot(\gamma_{km}, \frac{r}{k})$*, for each* $k \in \mathbb{N} \ni m$. *And the points* O, $\pi_{kj}^{m\pm}$*, for all* $k \in \mathbb{N}$*, are aligned with fixed sign* $\pm$, j*, and* m.

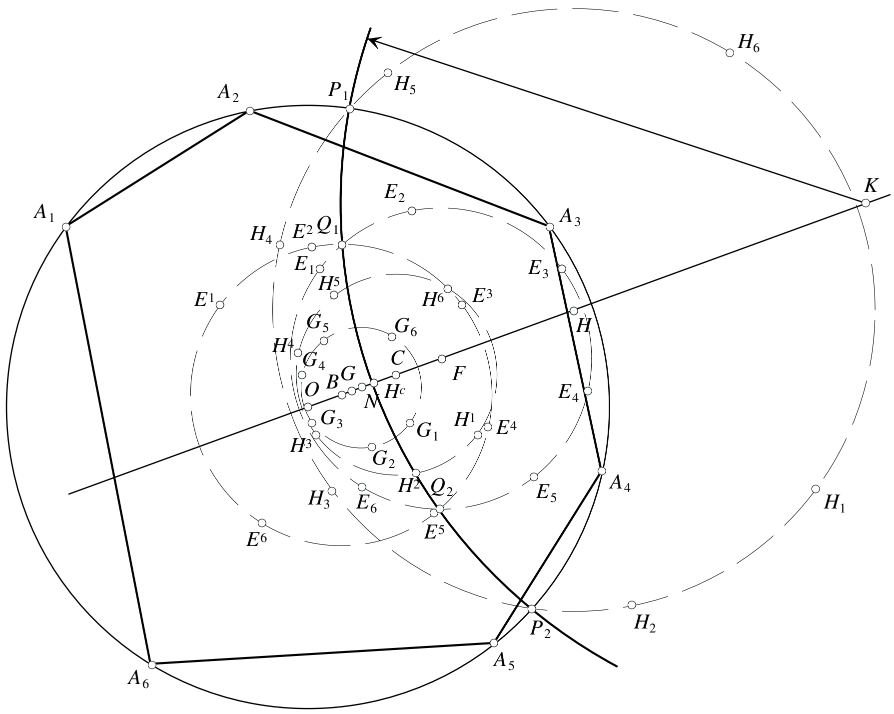

Fig. 4

5 Relation between the circles of radius r and $\frac{r}{2}$

To give a complete study, see Fig. 4, we consider the intersection of the circumcircle $\odot(O,r)$ and the remarkable circle $\odot(H,r)$: $\odot(H,r)\cap\odot(O,r) = \{P_1, P_2\}$. Also we consider the intersection of the Feuerbach circle $\odot(F,\frac{r}{2})$ and the Feuerbach-Collings circle $\odot(B,\frac{r}{2})$: $\odot(F,\frac{r}{2})\cap\odot(B,\frac{r}{2}) = \{Q_1, Q_2\}$. To make the calculations, we can consider $r = 1$. Then with $H = \left(\sum_{i=1}^{n}\cos\alpha_i, \sum_{i=1}^{n}\sin\alpha_i\right) = (a,b)$ we can make a rotation $g_{O,\varphi}$, of center O and angle φ, such that $g_{O,\varphi}(a,b) = (h,0)$. Then we have $g_{O,\varphi}(O) = O$, $g_{O,\varphi}(H) = (h,0)$, $g_{O,\varphi}(F) = \frac{1}{2}(h,0)$ and $g_{O,\varphi}(B) = \frac{1}{2n-4}(a,b)$, where $h = OH = \sqrt{a^2+b^2}$. Therefore, with a long and straightforward calculation we can proof the following:

Theorem 4 *Let $A_1A_2\ldots A_n$ be a cyclic n-polygon, $n\geq 4$, with circumradius r. Then:*

a) *The four points of intersection $\odot(H,r)\cap\odot(O,r)$ and $\odot(F,\frac{r}{2})\cap\odot(B,\frac{r}{2})$ are concyclic on the circle $\odot(K,\sigma)$.*

b) *The point K, on the Euler line, is $K = O + \frac{1}{2}\frac{3n-6-h^2}{h^2(n-3)}\overrightarrow{OH}$; where $h = OH$.*

c) *The radius $\sigma = \frac{r}{2h(n-3)}\sqrt{h^4(2n-5)+h^2(-2n^2+12)+9(n^2-4n+4)}$.*

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