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An explicit family of U_m-numbers

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1 Introduction

Transcendental number theory began in 1844 with Liouville's proof [7] that if an algebraic number α has degree $n > 1$, then there exists a constant $C > 0$ such that $|\alpha - p/q| > Cq^{-n}$, for all $p/q \in \mathbb{Q} \setminus \{0\}$. Using this result, Liouville gave the first explicit examples of transcendental numbers, the so-called Liouville numbers: a real number ξ is called a *Liouville number*, if for any positive real number ω there exist infinitely many rational numbers p/q, with $q \geq 1$, such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^{\omega}}.$$

Im Jahre 1844 gab Joseph Liouville ein erstes Beispiel einer transzendenten Zahl, nämlich die Liouville-Zahl $\ell = \sum_{n\geq 1} 10^{-n!}$. Fast ein Jahrhundert später schlug Kurt Mahler vor, die reellen Zahlen in vier Kategorien einzuteilen, je nachdem wie gut sie sich bei der Approximation durch algebraische Zahlen verhalten. Insbesondere zerfielen dabei die transzendenten Zahlen in drei Klassen, nämlich die S-, T- und U-Zahlen. 1952 bemerkte LeVeque, dass die U-Zahlen sich noch weiter, bezüglich ihrer Approximierbarkeit durch algebraische Zahlen vom Grad m, in die unendlich vielen disjunkten Klassen der U_m-Zahlen unterteilen lassen. LeVeque zeigte insbesondere, dass $\sqrt[m]{(3+\ell)/2}$ eine U_m-Zahl ist. In der vorliegenden Arbeit konstruieren die Autoren U_m-Zahlen für alle m auf besonders transparente Weise, nämlich als Produkt von ℓ mit gewissen algebraischen Zahlen vom Grad m.

A classical example of a Liouville number is the *Liouville constant* ℓ, defined as a decimal with a 1 in each decimal place corresponding to $n!$ and 0 otherwise. It can be represented by the fast convergent series $\ell = \sum_{n=1}^{\infty} 10^{-n!} = 0.1100010\ldots$.

In 1962, Erdős [4] proved that every nonzero real number can be written as the sum and the product of two Liouville numbers. Since the set of the Liouville numbers has null Lebesgue measure, one may interpret this as saying that in spite of being an "invisible" set, the Liouville numbers are strategically disposed along the real line.

There exist several classifications of the transcendental numbers in the literature. One attempt towards a classification was made in 1932 by Mahler [8], who proposed to subdivide the set of real numbers into four classes (one of them being the class of algebraic numbers) according to their properties of approximation by algebraic numbers. For instance, he split the set of transcendental numbers into three disjoint sets named *S-, T-* and *U-numbers*. Particularly, the U-numbers generalize the concept of Liouville numbers.

We denote by $\omega_n^*(\xi)$ as the supremum of the real numbers ω^* for which there exist infinitely many real algebraic numbers α of degree n satisfying

$$0 < |\xi - \alpha| < \mathcal{H}(\alpha)^{-\omega^* - 1},$$

where $\mathcal{H}(\alpha)$ (so-called the *height* of α) is the maximum of absolute values of coefficients of the minimal polynomial[1] of α. The number ξ is said to be a *U_m^*-number* (according to LeVeque [6]) if $\omega_m^*(\xi) = \infty$ and $\omega_n^*(\xi) < \infty$ for $1 \leq n < m$ (m is called the *type* of the U-number). We point out that we actually have defined a Koksma U_m^*-number instead of a Mahler U_m-number. However, it is well known that they are the same [3, cf. Theorem 3.6] and [1]. We remark that the set of U_1-numbers is precisely the set of Liouville numbers.

The existence of U_m-numbers for all $m \geq 1$, was first proved by LeVeque [6]. Indeed, he was able to exhibit such examples as the mth root of some convenient Liouville numbers, e.g., $\sqrt[m]{(3+\ell)/4}$ is a U_m-number, for all $m \geq 1$.

In this note, we use the Gütting method [5] to prove that we can find explicit U_m-numbers in a more natural way: the product of certain m-degree algebraic numbers by ℓ. Moreover, we obtain an upper bound for ω_n^*. More precisely, our result is the following

Theorem 1. *Let α be an algebraic number of degree m. Suppose that the minimal polynomial P of α has a leading coefficient of the form $2^a \cdot 5^b > 1$, and $p \nmid P(0)$, for $p = 2, 5$, and let ℓ be the Liouville constant. Then $\alpha\ell$ is a U_m-number, with*

$$\omega_n^*(\alpha\ell) \leq 2m^2 n + m - 1, \ \ for\ n = 1, \ldots, m-1. \tag{1}$$

For example, $\sqrt[m]{3/2} \cdot \ell$ is a U_m-number for all $m \geq 1$.

[1] Throughout the paper, a polynomial is said to be minimal if it is a primitive minimal polynomial over $\mathbb{Z}$.

2 Auxiliary Results

Before starting the proof of the Theorem, two technical results are needed.

Lemma 1. *Given $P(x) \in \mathbb{Z}[x]$ with degree m and $a/b \in \mathbb{Q}\backslash\{0\}$. If $Q(x) = a^m P(bx/a)$, then*

$$\mathcal{H}(Q) \le \max\{|a|, |b|\}^m \mathcal{H}(P),$$

where, as usual, $\mathcal{H}(P)$ denotes the maximum of absolute values of coefficients of P (the so-called height *of P).*

Proof. If $P(x) = \sum_{j=0}^{m} a_j x^j$, then $Q(x) = \sum_{j=0}^{m} a_j b^j a^{m-j} x^j$. Supposing, without loss of generality, that $|a| \ge |b|$, we have $|a|^m |a_j| \ge |a|^{m-j} |a_j| |b|^j$ for $0 \le j \le m$. Hence, we are done. □

In addition to Lemma 1, we use the fact that algebraic numbers are not well approximable by algebraic numbers.

Lemma 2 (Cf. Corollary A.2 of [3])**.** *Let α and β be two distinct nonzero algebraic numbers of degree n and m, respectively. Then we have*

$$|\alpha - \beta| \ge (n+1)^{-m/2}(m+1)^{-n/2} \max\left\{\frac{(n+1)^{-(m-1)/2}}{2^{-n}}, \frac{(m+1)^{-(n-1)/2}}{2^{-m}}\right\}$$
$$\times\, H(\alpha)^{-m} H(\beta)^{-n}.$$

Proof. A sketch of the proof can be found in Appendix A of [3]. □

3 Proof of the Theorem

For $k \ge 1$, set

$$p_k = 10^{k!} \sum_{j=1}^{k} 10^{-j!}, \quad q_k = 10^{k!} \quad \text{and} \quad \alpha_k = \frac{p_k}{q_k}.$$

We observe that $\mathcal{H}(\alpha_{k-1}) < \mathcal{H}(\alpha_k) = 10^{k!} = \mathcal{H}(\alpha_{k-1})^k$ and

$$|\ell - \alpha_k| < \frac{10}{9} \mathcal{H}(\alpha_k)^{-k-1}. \tag{2}$$

Thus, setting $\gamma_k = \alpha\alpha_k$, we obtain of (2)

$$|\alpha\ell - \gamma_k| \le c\mathcal{H}(\alpha_k)^{-k-1}, \tag{3}$$

where $c = 10|\alpha|/9$. It follows by Lemma 1 that $\mathcal{H}(\alpha_k)^m \ge \mathcal{H}(\alpha)^{-1}\mathcal{H}(\gamma_k)$ and thus we conclude that

$$|\alpha\ell - \gamma_k| \le c\mathcal{H}(\alpha)^{(k+1)/m} \mathcal{H}(\gamma_k)^{-(k+1)/m}. \tag{4}$$

Consequently, $\alpha\ell$ is a U-number with type at most m (since γ_k has degree m).

We claim that $H(\alpha_k) \leq H(\gamma_k)$, for all $k \geq 1$. In fact, let $P(x) = \sum_{j=0}^{m} a_j x^j$ be the minimal polynomial of α. In particular, $P(\alpha) = 0$ and a simple calculation gives $Q(\gamma_k) = 0$, where $Q(x) = \sum_{j=0}^{m} a_j p_k^{m-j} q_k^j x^j \in \mathbb{Z}[x]$. Note that $\deg Q = m$ and γ_k is an m-degree algebraic number. Thus, in order to prove that Q is the minimal polynomial of γ_k, we need to prove that Q is primitive. In other words, we must prove that

$$\gcd(a_0 p_k^m, a_1 p_k^{m-1} q_k, \ldots, a_m q_k^m) = 1.$$

This follows immediately from the facts that $\gcd(a_0, \ldots, a_m) = 1$ and the hypotheses on a_0 and a_m (yielding $\gcd(a_0, q_k) = \gcd(a_m, p_k) = 1$), we leave the details to the reader. Thus, in particular, we have that

$$H(\gamma_k) \geq \max\{|a_0||p_k|^n, |a_n||q_k|^n\} \geq \max\{|p_k|, |q_k|\} = H(\alpha_k)$$

as desired.

Now we use this together with Lemma 1 to obtain

$$\mathcal{H}(\gamma_{k+1}) \leq \mathcal{H}(\alpha)\mathcal{H}(\alpha_{k+1})^m = \mathcal{H}(\alpha)\mathcal{H}(\alpha_k)^{(k+1)m} \leq \mathcal{H}(\alpha)\mathcal{H}(\gamma_k)^{(k+1)m}. \tag{5}$$

Now, let γ be an n-degree real algebraic number, with $n < m$ and $\mathcal{H}(\gamma) \geq \mathcal{H}(\gamma_1)$. Thus, there exists a sufficient large k such that

$$\mathcal{H}(\gamma_k) < \mathcal{H}(\gamma)^{2m^2} < \mathcal{H}(\gamma_{k+1}) \leq \mathcal{H}(\alpha)\mathcal{H}(\gamma_k)^{(k+1)m}. \tag{6}$$

On the other hand, Lemma 2 yields

$$|\gamma_k - \gamma| \geq f(m,n)\mathcal{H}(\gamma)^{-m}\mathcal{H}(\gamma_k)^{-n}, \tag{7}$$

where $f(m,n)$ is a positive number which does not depend on k and γ (see Lemma 2). Therefore by the chain of inequalities in (6)

$$|\gamma_k - \gamma| \geq f(m,n)\mathcal{H}(\alpha)^{-1/2m}\mathcal{H}(\gamma_k)^{-(k+1)/2-n}. \tag{8}$$

By taking $\mathcal{H}(\gamma)$ large enough, the index k satisfies

$$\mathcal{H}(\gamma_k)^{(k+1)/2-n} \geq 2cf(m,n)^{-1}\mathcal{H}(\alpha)^{k+1/2m}. \tag{9}$$

Thus (4), (8) and (9) yield that $|\gamma_k - \gamma| \geq 2|\alpha\ell - \gamma_k|$. Therefore, for all n-degree algebraic numbers with a sufficiently large weight, we have

$$\begin{aligned} |\alpha\ell - \gamma| &\geq |\gamma_k - \gamma| - |\alpha\ell - \gamma_k| \geq \frac{1}{2}|\gamma_k - \gamma| \\ &\geq \frac{f(m,n)}{2}\mathcal{H}(\gamma)^{-m}\mathcal{H}(\gamma_k)^{-n} > \frac{f(m,n)}{2}\mathcal{H}(\gamma)^{-2m^2n-m}, \end{aligned}$$

where we used the left-hand side of (6). In conclusion, $\alpha\ell$ is a U_m-number with $\omega_n^*(\alpha\ell) \leq 2m^2n + m - 1$. This finishes the proof. □

We finish by pointing out that Alniaçik et al. [2] showed the existence of U_m-numbers ξ with sharper upper bounds for $\omega_n^*(\xi)$, where $n = 1, \ldots, m-1$. However, in their method ξ is constructed as the limit of a rapidly converging sequence of m-degree algebraic numbers and therefore could not be made explicit.

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