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Elemente der Mathematik

Inequalities comparing $(a+b)^p - a^p - b^p$ and $a^{p-1}b + ab^{p-1}$

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Consider the comparison between $(a+b)^p$ and $a^p + b^p$, where a, b and p are positive. It is elementary that $(a+b)^p > a^p + b^p$ for $p > 1$ and the opposite holds for $0 < p < 1$ (let $b/a = x \leq 1$: then for $p > 1$, we have $(1+x)^p > 1 + x > 1 + x^p$). Let us write

$$F_p(a, b) = (a+b)^p - a^p - b^p.$$

For $p = 2,\ 3$, we have the identities

$$F_2(a, b) = 2ab, \qquad F_3(a, b) = 3(a^2 b + ab^2).$$

Also, when b/a is small, $(a+b)^p$ is approximated by $a^p + pa^{p-1}b$. These facts suggest that it is a natural idea to look for estimates of $F_p(a, b)$ in terms of

$$G_p(a, b) = a^{p-1}b + ab^{p-1}.$$

Here we will seek to determine, for each $p > 0$, the best constants A_p, B_p such that

$$A_p G_p(a, b) \leq F_p(a, b) \leq B_p G_p(a, b) \tag{1}$$

In der Funktionalanalysis oder bei der Untersuchung nichtlinearer partieller Differentialgleichungen spielen oft elementare Ungleichungen im Zusammenhang mit p-ten Potenzen von Termen eine Rolle. Bekannt ist etwa die Ungleichung von Clarkson. Der Autor der vorliegenden Arbeit geht aus von der Ungleichung $(a+b)^p > a^p + b^p$ für positive Zahlen a, b und $p > 1$. Für $0 < p < 1$ gilt just die umgekehrte Ungleichung. Untersucht wird nun der Defekt $(a+b)^p - a^p - b^p =: F_p(a, b)$. Da $F_2(a, b) = 2ab$ und $F_3(a, b) = 3(a^2b + ab^2)$ gilt, liegt es nahe, $F_p(a, b)$ durch $G_p(a, b) = a^{p-1}b + ab^{p-1}$ abzuschätzen. Es stellt sich heraus, dass die bestmöglichen Konstanten A_p und B_p in der Ungleichung $A_p G_p(a, b) \leq F_p(a, b) \leq B_p G_p(a, b)$ in erstaunlich verwickelter Weise von p abhängen.

for all $a, b > 0$. As we shall see, it is quite easy to establish a version of the right-hand inequality with non-optimal B_p; weak upper estimates of this kind have undoubtedly been stated and used many times. However, lower estimates are less well known, and despite the wealth of known results on sums of pth powers, (e.g., [1, 2, 3] and numerous research articles), I am not aware of any consideration of the best constants in the existing literature. The solution, given in Theorem 1 below, turns out to be surprisingly intricate, with A_p and B_p switching between different expressions at the values 1, 2 and 3 of p, in a way that indicates that upper and lower bounds may emerge from the same process of reasoning.

The problem is reduced to a single-variable one by the substitution $x = b/a$: it is easily checked, on dividing by a^p, that (1) is equivalent to

$$A_p g_p(x) \le f_p(x) \le B_p g_p(x) \tag{2}$$

for $x > 0$, where

$$f_p(x) = (1+x)^p - 1 - x^p, \qquad g_p(x) = x + x^{p-1}.$$

Write also $h_p(x) = f_p(x)/g_p(x)$. Clearly, $h_p(1) = 2^{p-1} - 1$: This quantity will play an important part in our considerations: we denote it by C_p. So certainly we have $A_p \le C_p \le B_p$. Also, for all $x > 0$, we have $h_1(x) = 0$, $h_2(x) = 1$ and $h_3(x) = 3$.

Lemma 1. *We have $h_p(1/x) = h_p(x)$, hence if* (2) *holds* (*for a certain A_p, B_p*) *for $0 < x \le 1$* (*or for $x \ge 1$*)*, then it holds for all $x > 0$.*

Proof. Clearly, $f_p(x) = x^p f_p(1/x)$, and similarly for g_p. □

Lemma 2. *We have*

$$\lim_{x\to 0^+} h_p(x) = \begin{cases} p & \text{if } p > 2, \\ 1 & \text{if } p = 2, \\ 0 & \text{if } 0 < p < 2. \end{cases}$$

Proof. The cases $p = 1, 2$ are trivial. Let $p > 2$. Then $f_p(0) = g_p(0) = 0$, also $f_p'(0) = p$ and $g_p'(0) = 1$. By L'Hôpital's rule, $\lim_{x\to 0^+} h_p(x) = p$.

Next, let $1 < p < 2$. We still have $f_p'(0) = p$, hence $f_p(x)/x \to p$ as $x \to 0^+$. Also, $g_p(x) > x^{p-1}$, so $x/g_p(x) < x^{2-p} \to 0$ as $x \to 0^+$. Hence $h_p(x) \to 0$ as $x \to 0^+$.

Finally, let $0 < p < 1$. Then $-x^p < f_p(x) < 0$ and $g_p(x) > x^{p-1}$, so $|h_p(x)| < x$. □

Before dealing with the general case, we show that there is a quick solution to our problem for integer values of p:

Proposition 1. *For integers $p \ge 3$, the best constants in* (1) *and* (2) *are: $A_p = p$, $B_p = C_p$.*

Proof. By Lemma 1, it is sufficient to consider (2) with $0 \le x \le 1$. By adding together two copies of the binomial expansion, we have

$$2f_p(x) = \sum_{r=1}^{p-1} \binom{p}{r} (x^r + x^{p-r}).$$

For $2 \le r \le p-2$ and $0 \le x \le 1$, we have

$$(x + x^{p-1}) - (x^r + x^{p-r}) = (1 - x^{r-1})(x - x^{p-r}) \ge 0.$$

Hence

$$2f_p(x) \le \sum_{r=1}^{p-1} \binom{p}{r}(x + x^{p-1}) = (2^p - 2)(x + x^{p-1}),$$

so $f_p(x) \le C_p g_p(x)$. As we have seen, equality occurs when $x = 1$. It is also clear from the binomial expansion that $f_p(x) \ge p(x + x^{p-1}) = pg_p(x)$, and Lemma 2 shows that p is the best constant in this inequality. □

We now reveal the full solution to our problem. It is rather more interesting than one might have expected in the light of the previous result.

Theorem 1. *The best constants* A_p, B_p *in* (1) *and* (2) *are as follows:*

p	A_p	B_p
$[3, \infty)$	p	C_p
$(2, 3)$	C_p	p
2	1	1
$(1, 2)$	0	C_p
$(0, 1]$	C_p	0

Before giving the proof, we record some comments on the result.

(1) The reversals at 1, 2 and 3 are not altogether surprising, given that $h_p(x)$ is constant for these values of p.

(2) When $0 < p < 1$, both $f_p(x)$ and C_p are negative.

(3) At $p = 2$, A_p is discontinuous from below and B_p discontinuous from above, reflecting the discontinuity in Lemma 2.

(4) The statement incorporates the fact, not instantly transparent, that $C_p \ge p$ for $p \ge 3$ and $C_p \le p$ for $2 \le p \le 3$. To see this directly, note that C_p is a convex function of p and $C_2 = 1$, $C_3 = 3$. The linear function interpolating these two values is $h(p) = 2p - 3$, so for $2 \le p \le 3$, we have $C_p \le 2p - 3 \le p$, while for $p \ge 3$, we have $C_p \ge 2p - 3 \ge p$.

Lemma 3. *If f is convex on $[0, \infty)$ and $f(0) = 0$, then $f(x)/x$ is increasing for $x > 0$. If f is concave, then $f(x)/x$ is decreasing.*

Proof. Assume that f is convex. Let $0 < x < y$ and write $\lambda = x/y$. Then $x = (1 - \lambda)0 + \lambda y$, so $f(x) \le (1 - \lambda)f(0) + \lambda f(y) = \lambda f(y)$, hence $f(x)/x \le f(y)/y$. □

Lemma 4. *Let $0 \le x \le 1$. Then $(1 + x)^p - 1 \le (2^p - 1)x$ for $p \ge 1$ and $p \le 0$, and the reverse inequality holds for $0 \le p \le 1$.*

Proof. Let $f(x) = (1 + x)^p - 1$. For $p \ge 1$ and $p \le 0$, f is convex, so by Lemma 3, $f(x)/x \le f(1) = 2^p - 1$. For $0 < p < 1$, f is concave, so the reverse inequality holds. □

With a term discarded on each side, Lemma 4 implies that $f_p(x) \le (2^p - 1)g_p(x)$ for all $p > 1$. So it gives a (very quick) proof that $B_p \le 2^p - 1$ for such p. This weaker version is surely well known, and adequate for some applications.

Before continuing with the proof of Theorem 1, we digress briefly to show that another application of Lemma 3 gives a complete solution to the following natural variant of the original problem.

Suppose that we stipulate that $a \geq b$ and seek to compare $F_p(a,b)$ with the term $a^{p-1}b$ on its own. In other words, we look for the best constants D_p, E_p such that

$$D_p a^{p-1} b \leq F_p(a,b) \leq E_p a^{p-1} b \tag{3}$$

for $a \geq b > 0$. Equivalently, $D_p x \leq f_p(x) \leq E_p x$ for $0 \leq x \leq 1$.

Proposition 2. *For $p \geq 2$, we have $D_p = p$ and $E_p = 2^p - 2$ $(= 2C_p)$. For $1 < p \leq 2$, these two values are reversed.*

Proof. Apply Lemma 3 to $f_p(x)$. For $p \geq 2$ and $x > 0$,

$$f_p''(x) = p(p-1)(1+x)^{p-2} - p(p-1)x^{p-2} \geq 0,$$

so f_p is convex. By Lemma 3, $f_p(x)/x$ is increasing. So its greatest value on $(0,1]$ is $f_p(1) = 2^p - 2$, and its infimum is $\lim_{x \to 0^+} f_p(x)/x = f_p'(0) = p$. For $1 < p < 2$, f_p is concave, so the two bounds are interchanged. □

Note. When $1 < p < 2$, x^{p-1} is larger than x, so it is really more natural to compare $f_p(x)$ with x^{p-1}. We saw in Lemma 2 that $\inf_{x>0}[f_p(x)/x^{p-1}] = 0$ for such p. For the upper bound, note that since $f_p(x)/x$ is decreasing, we have $f_p(x) \leq (2^p - 2)x$ for $x \geq 1$. Substituting $1/x$ for x, we deduce that $f_p(x) \leq (2^p - 2)x^{p-1}$ for $0 < x \leq 1$.

We return to the proof of Theorem 1.

Lemma 5. *Let $\phi(p) = p2^{p-2} - 2^p + 2$. Then $\phi(p) \geq 0$ for $p \geq 3$ and $p \leq 2$, and $\phi(p) \leq 0$ for $2 \leq p \leq 3$.*

Proof. Let $\psi(p) = 2^{2-p}\phi(p) = p - 4 + 2^{3-p}$. Then $\psi(p)$ is a convex function of p, and $\psi(2) = \psi(3) = 0$. Hence $\psi(p) \geq 0$ for $p \geq 3$ and $p \leq 2$, and $\psi(p) \leq 0$ for $2 \leq p \leq 3$. □

Proof of Theorem 1. First, it follows from our opening observations and Lemma 2 that $A_p = 0$ for $1 < p < 2$ and $B_p = 0$ for $0 < p < 1$.

Next, let $\Phi_p(x) = f_p(x) - pg_p(x)$. We show that for all $x > 0$, $\Phi_p(x) \geq 0$ if $p \geq 3$ and $\Phi_p(x) \leq 0$ if $2 < p \leq 3$. With Lemma 2, it then follows that $A_p = p$ for $p \geq 3$ and $B_p = p$ for $2 < p \leq 3$. Since $\Phi_p(0) = 0$, these inequalities will follow if similar ones are satisfied by $\Phi_p'(x)$. Now

$$\frac{1}{p}\Phi_p'(x) = (1+x)^{p-1} - x^{p-1} - 1 - (p-1)x^{p-2}.$$

For $p > 2$, $\Phi_p'(0) = 0$. We proceed to the second derivative and reason similarly:

$$\begin{aligned}\frac{1}{p(p-1)}\Phi_p''(x) &= (1+x)^{p-2} - x^{p-2} - (p-2)x^{p-3} \\ &= x^{p-2}\left[\left(1+\frac{1}{x}\right)^{p-2} - 1 - \frac{p-2}{x}\right].\end{aligned}$$

It is well known that $(1+y)^q \geq 1+qy$ for $y > 0$ if $q \geq 1$ and the reverse inequality holds if $0 < q < 1$. Hence $\Phi_p''(x) \geq 0$ if $p \geq 3$, and $\Phi_p''(x) \leq 0$ if $2 < p \leq 3$, so similar inequalities are satisfied by $\Phi_p'(x)$ and $\Phi_p(x)$, as required.

Now let $\Psi_p(x) = f_p(x) - C_p g_p(x)$. We show that for all $x > 0$, $\Psi_p(x) \leq 0$ for p in $[3, \infty)$ and $(1, 2)$ and $\Psi_p(x) \geq 0$ for p in $(2, 3)$ and $(0, 1)$. Since $f_p(1) = C_p g_p(1)$, it then follows that $B_p = C_p$ in the first two cases and $A_p = C_p$ in the second two. By Lemma 1, it is sufficient to prove the stated inequalities for $x \geq 1$, and since $\Psi_p(1) = 0$, it is enough to prove similar inequalities for $\Psi_p'(x)$. (Note that no such statement applies for $0 < x < 1$, since Ψ_p is zero at 0 and 1.) Now

$$\Psi_p'(x) = p(1+x)^{p-1} - px^{p-1} - C_p[1 + (p-1)x^{p-2}].$$

In particular, $\Psi_p'(1) = p2^{p-1} - p - pC_p = 0$. Again we proceed to the second derivative:

$$\frac{1}{p-1}\Psi_p''(x) = p(1+x)^{p-2} - px^{p-2} - (p-2)C_p x^{p-3}.$$

Write this as $x^{p-2}S_p(x)$, where

$$S_p(x) = p\left(1 + \frac{1}{x}\right)^{p-2} - p - \frac{(p-2)C_p}{x}.$$

Now write $y = 1/x$, so $0 < y \leq 1$. By Lemma 4, for $p \geq 3$ and $p < 2$,

$$\begin{aligned} S_p(x) &= p\big[(1+y)^{p-2} - 1\big] - (p-2)C_p y \\ &\leq \big[p(2^{p-2} - 1) - (p-2)C_p\big]y. \end{aligned}$$

By Lemma 5, we have, again for $p \geq 3$ and $p < 2$,

$$p(2^{p-2} - 1) - (p-2)C_p = 2^p - 2 - p2^{p-2} \leq 0,$$

Hence $\Psi_p''(x) \leq 0$ for $p \geq 3$ and $1 < p < 2$, while $\Psi_p''(x) \geq 0$ for $0 < p < 1$, because of the factor $p-1$. For $2 < p < 3$, both inequalities reverse, giving $\Psi_p''(x) \geq 0$. Our proof is complete. □

Question. Is $h_p(x)$ increasing on $(0, 1]$ for p in $(3, \infty)$ and $(1, 2)$, and decreasing for p in $(2, 3)$ and $(0, 1)$? Theorem 1 would, of course, be an immediate consequence. The following remark may illuminate this question a little. As we saw in Proposition 2, $x/f_p(x)$ is decreasing on $(0, \infty)$ for $p \geq 2$ and increasing for $1 \leq p \leq 2$. By the substitution $y = 1/x$, we deduce that $x^{p-1}/f_p(x)$ does the opposite. So in this way, $1/h_p(x)$ is expressed as the sum of two functions, one increasing and one decreasing, which suggests that the question of its monotonicity is more delicate.

An application to the Banach spaces ℓ_p. For $p \geq 1$, the (real) Banach sequence space ℓ_p is the space of infinite real sequences $x = (x_n)$ such that $\sum_{n=1}^{\infty} |x_n|^p$ convergent (say to $N_p(x)$), with the norm $\|x\|_p = N_p(x)^{1/p}$. For non-negative sequences a, b in ℓ_p, we clearly have $\|a+b\|_p^p \geq \|a\|_p^p + \|b\|_p^p$, with equality occurring when a and b are "orthogonal" in the sense that for each n, we have $a_n b_n = 0$ (so that either a_n or b_n is 0). Equality also occurs when $p = 1$. Note that $\|a+b\|_p^p - \|a\|_p^p - \|b\|_p^p = \sum_{n=1}^{\infty} F_p(a_n, b_n)$. Theorem 1 translates into the following estimation of how close we are to equality when a and b are nearly orthogonal, in the sense that all the products $a_n b_n$ are small, or when p is close to 1.

Proposition 3. *Let a, b be non-negative elements of ℓ_p, and (by extension of our previous notation), let*

$$G_p(a,b) = \sum_{n=1}^{\infty} (a_n^{p-1} b_n + a_n b_n^{p-1}).$$

Then, with A_p, B_p as given in Theorem 1, we have

$$A_p G_p(a,b) \leq \|a+b\|_p^p - \|a\|_p^p - \|b\|_p^p \leq B_p G_p(a,b).$$

We can derive a second, more specific, result of this type from Proposition 2.

Proposition 4. *Let a, b be non-negative elements of ℓ_p and let $\delta \in [0,1]$ be such that for each n, either $b_n \leq \delta a_n$ or $a_n \leq \delta b_n$. Then*

$$\|a+b\|_p^p \leq (1 + \delta E_p)(\|a\|_p^p + \|b\|_p^p),$$

where

$$E_p = \begin{cases} 2^p - 2 & \text{for } p \geq 2, \\ p & \text{for } 1 \leq p \leq 2. \end{cases}$$

Proof. Let $N_1 = \{n : a_n \geq b_n\}$ and $N_2 = \{n : b_n > a_n\}$. By Proposition 2, we have for $n \in N_1$

$$F_p(a_n, b_n) \leq E_p a_n^{p-1} b_n \leq \delta E_p a_n^p.$$

Hence $\sum_{n \in N_1} F_p(a_n, b_n) \leq \delta E_p \sum_{n \in N_1} a_n^p \leq \delta E_p \sum_{n=1}^{\infty} a_n^p$. Adding a similar estimate for N_2, we obtain our statement. □

Note. In the case $1 \leq p \leq 2$, by applying the note on Proposition 2 and comparing with $a_n b_n^{p-1}$, we obtain an alternative estimation with δp replaced by $(2^p - 2)\delta^{p-1}$. This is not always stronger, but it gives the correct value 0 when $p = 1$.

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