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Linear functional equations involving Babbage's equation*

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1 Introduction

A functional equation is an equation whose unknowns are functions. Cauchy's functional equation [17] $\varphi(x + y) = \varphi(x) + \varphi(y)$, Schröder's equation [28, 39] $\varphi \circ f = s\varphi$, and Schilling's equation [7, 25] $4q\varphi(qx) = \varphi(x+1)+2\varphi(x)+\varphi(x-1)$ are examples of such equations.

Functional equations arise in many branches of mathematics, for example, dynamical systems [1, 19, 24, 43], functional analysis [42], geometry [8, 9], information theory [3], wavelet theory [20, 21], and special functions [27]. They also occur in other disciplines such as physics [22, 33], engineering [15, 16], economics [4, 23] and so on.

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Funktionalgleichungen bilden nicht nur ein reichhaltiges Forschungsthema, sondern sie sind auch beliebte Probleme bei Mathematikwettbewerben. Oft entspringen Funktionalgleichungen konkreten Anwendungen. Sucht man etwa für ein diskretes dynamisches System $x \mapsto f(x)$ ein erstes Integral ϕ, so entspricht dies gerade dem Auffinden einer nicht konstanten Lösung der Funktionalgleichung $\varphi \circ f = \varphi$. Die Autoren untersuchen in ihrer Arbeit eine Klasse von Funktionalgleichungen, welche mit der Babbage-Gleichung in Beziehung steht. Letztere fragt nach einer Funktion f, deren n-te Iterierte f^n die Identität ist. Insbesondere werden in der vorliegenden Arbeit explizite Lösungen für die Gleichung $\varphi = \pm\varphi \circ f + g$ angegeben, wobei f eine Lösung der Babbage-Gleichung, und g eine gegebene Funktion ist.

The systematic study of functional equations did not begin until 1966 [2], although many great mathematicians have been studying them before, including Euler (1768), Cauchy (1821), Abel (1823), Darboux (1895), and Banach (1920) (cf. [27]). In the last five decades, the theory of functional equations has developed very rapidly and gradually became an independent field of mathematics. Functional equations also became a common topic in mathematics competitions, see the books [13, 30, 41], some problems and solutions in the journals *The American Mathematical Monthly* and *Mathematical Excalibur* [18], and the website “KöMaL” [34].

Apart from competition problems, a considerable number of interesting problems (see, e.g., [10, 14, 24]) involve the following single variable functional equation – Babbage’s equation

$$\varphi^n = \mathrm{id}, \tag{1}$$

where φ^n denotes the nth iterate of a self-map φ, and id stands for the identity. Ch. Babbage [5, 6] studied its solutions in the reals. In 1916, J.F. Ritt [38] gave four types of real solutions. Later, the results on Babbage’s equation were generalized into many different directions, e.g., continuous solutions in [28, Theorem 15.2], meromorphic solutions in [28, pp. 291–292], also [40, Example 2], homeomorphic solutions on the unit circle in [26], and involutions on the plane in [31].

Motivated by the functional equation $\varphi \circ f = \varphi$ for an integrable map f (see [24]) and the competition problem to determine the function $\varphi : \mathbb{R}\setminus\{0, 1\} \to \mathbb{R}$ such that

$$\varphi(x) + \varphi\left(\frac{x-1}{x}\right) = 1 + x,$$

this paper investigates the single variable functional equations

$$\varphi = \pm\varphi \circ f + g, \tag{2}$$

where f, g are given and f is globally periodic with the prime period n (i.e., $f^i \neq \mathrm{id}$ for $1 < i < n$ and $f^n = \mathrm{id}$).

The general form of these equations above is

$$F(\varphi \circ f_1, \ldots, \varphi \circ f_n, \mathrm{id}) = 0, \tag{3}$$

where $F, f_1, \ldots, f_n$ are given and φ is unknown. When F is linear and the functions $f_1, \ldots, f_n$ form a group under composition on their domain, S. Presić [29, 36, 37] characterized all solutions of (3). The unique solution of a special case in (3) is determined by M. Bessenyei [10] under additional regularity assumptions. Further investigations have been carried out by M. Bessenyei and his collaborators [11, 12] for the unique differentiable solution of (3).

The equation (2) is another special case of (3). With the methods of linear algebra combined with a version of recurrent iteration, we present exact solutions of (2) and the formulas of solutions are different from those in [32]. We also present some examples and applications.

2 The main results

The equation (2) is a class of linear functional equations and the corresponding homogeneous equation is

$$\varphi = \pm\varphi \circ f. \tag{4}$$

Similar to [29, p. 101, Theorem 3.1.5], we have the superposition principle for the linear functional equation (2).

Lemma 1. *Let S be a set and $(G, +)$ a group, $f : S \to S$ and $g : S \to G$ be two given mappings. Then the general solution $\varphi : S \to G$ of equation* (2) *is given by $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 : S \to G$ is a particular solution of* (2)*, and $\varphi_2 : S \to G$ is the general solution of equation* (4)*.*

Proof. Let $\varphi : S \to G$ be an arbitrary solution of (2) and $\varphi_1 : S \to G$ a particular solution of (2). Then

$$\varphi = \pm\varphi \circ f + g,$$
$$\varphi_1 = \pm\varphi_1 \circ f + g.$$

Thus $(\varphi - \varphi_1) = \pm(\varphi - \varphi_1) \circ f$. It follows that $\varphi - \varphi_1$ is a solution of (4).
On the other hand, let $\varphi_2 : S \to G$ be an arbitrary solution of (4) and $\varphi_1 : S \to G$ a particular solution of (2). Then

$$\varphi_2 = \pm\varphi_2 \circ f,$$
$$\varphi_1 = \pm\varphi_1 \circ f + g.$$

Thus $(\varphi_1 + \varphi_2) = \pm(\varphi_1 + \varphi_2) \circ f + g$. It follows that $\varphi_1 + \varphi_2$ is a solution of (2). □

In what follows, it suffices to find the general solution of the homogeneous equation (4) and one particular solution of (2).

Lemma 2. *Suppose f is globally periodic with the prime period n on a set S and the unknown φ maps the set S to a set G. Then the general solution of $\varphi = \varphi \circ f$ is given by*

$$\varphi(x) = H\left(x, f(x), f^2(x), \ldots, f^{n-1}(x)\right),$$

where $H : S^n \to G$ is any function satisfying

$$H\left(x, f(x), \ldots, f^{n-1}(x)\right) = H\left(f(x), f^2(x), \ldots, f^{n-1}(x), x\right).$$

Proof. Let $\varphi : S \to G$ be a solution of $\varphi = \varphi \circ f$. Then define $H : S^n \to G$ in this way: take an arbitrary $x_0 \in S$,

$$H\left(x_0, f(x_0), \ldots, f^{n-1}(x_0)\right) := \varphi(x_0), \quad \forall x_0 \in S;$$

on other points $(x_1, x_2, \ldots, x_n) \in S^n$, define H arbitrarily. We see that $C_f(x_0) := \{x_0, f(x_0), \ldots, f^{n-1}(x_0)\}$ is an orbit of x_0. It follows from [28, Theorem 1.6] that φ is constant on $C_f(x_0)$. So

$$H\left(x_0, f(x_0), \ldots, f^{n-1}(x_0)\right) = \varphi(x_0) = \varphi(f(x_0)) = H\left(f(x_0), \ldots, f^{n-1}(x_0), x_0\right).$$

On the other hand, a simple calculation shows that $\varphi := H$ satisfies $\varphi = \varphi \circ f$. □

Let n be an integer greater than or equal to 2. A *uniquely n-divisible Abelian group* $(K, +)$ is an Abelian group having the property that for each $x \in K$ there is a unique $y \in K$ such that $x = ny$. So we can denote y by $\frac{x}{n}$.

Lemma 3. *Suppose f is globally periodic with the prime period n on a set S, and $(G, +)$ is a uniquely n-divisible Abelian group. Then the general solution $\varphi : S \to G$ of $\varphi = \varphi \circ f$ is given by*

$$\varphi(x) = \sum_{i=0}^{n-1} h\left(f^i(x)\right), \tag{5}$$

where $h : S \to G$ is an arbitrary function.

Proof. For an arbitrary function $h : S \to G$, the function

$$\varphi(x) := \sum_{i=0}^{n-1} h\left(f^i(x)\right)$$

evidently satisfies $\varphi = \varphi \circ f$. On the other hand, if φ is a solution of the equation $\varphi = \varphi \circ f$, then $\varphi = \varphi \circ f^i$ for every positive integer i. Since $(G, +)$ is a uniquely n-divisible Abelian group, we have for any $x \in S$

$$\begin{aligned}\varphi(x) &= \frac{\varphi(x)}{n} + \frac{\varphi(f(x))}{n} + \cdots + \frac{\varphi(f^{n-1}(x))}{n}\\ &= \sum_{i=0}^{n-1} \frac{\varphi\left(f^i(x)\right)}{n}.\end{aligned}$$

Set $h(x) := \frac{\varphi(x)}{n}$. Then (5) holds. □

Lemma 4. *Suppose f is globally periodic with the prime period n on a set S, n is odd, and $(G, +)$ is a group and for each $y \in G$, $2y = 0$ if and only if $y = 0$. Then $\varphi = -\varphi \circ f$ has a unique solution from S to G given by $\varphi(x) = 0$.*

Proof. By successively substituting $f^j(x)$ for x in $\varphi(x) = -\varphi \circ f(x)$ for each $j = 1, 2, \ldots, n-1$, we obtain a set of n equations in the n unknowns $\varphi(f^j(x))$:

$$\begin{cases}\varphi(x) + \varphi(f(x)) = 0,\\ \varphi(f(x)) + \varphi(f^2(x)) = 0,\\ \vdots\\ \varphi(f^{n-2}(x)) + \varphi(f^{n-1}(x)) = 0,\\ \varphi(f^{n-1}(x)) + \varphi(x) = 0.\end{cases} \tag{6}$$

Since n is odd, we have

$$\varphi(x) = -\varphi(f(x)) = \varphi(f^2(x)) = \cdots = \varphi(f^{n-1}(x)) = -\varphi(x).$$

Thus $\varphi(x) = 0$. □

With similar arguments as in Lemmas 2, 3, proofs of the following two lemmas are easily supplied.

Lemma 5. *Suppose f is globally periodic with the prime period n on a set S, n is even, and $(G, +)$ is a group. Then the general solution $\varphi : S \to G$ of $\varphi = -\varphi \circ f$ is given by*

$$\varphi(x) = H\left(x, f(x), \ldots, f^{n-1}(x)\right),$$

where $H : S^n \to G$ is any function satisfying

$$H\left(x, f(x), \ldots, f^{n-1}(x)\right) + H\left(f(x), f^2(x), \ldots, f^{n-1}(x), x\right) = 0.$$

Lemma 6. *Suppose f is globally periodic with the prime period n on a set S, n is even, and $(G, +)$ is a uniquely n-divisible Abelian group. Then the general solution $\varphi : S \to G$ of $\varphi = -\varphi \circ f$ is given by*

$$\varphi(x) = \sum_{i=0}^{n-1} (-1)^i h\left(f^i(x)\right),$$

where $h : S \to G$ is an arbitrary function.

Now we shall give exact solutions of (2).

Theorem 1. *Suppose f is globally periodic with the prime period n on a set S, and $(G, +)$ is a uniquely n-divisible Abelian group. Then there exists a solution $\varphi : S \to G$ of $\varphi = \varphi \circ f + g$ if and only if $\sum_{i=0}^{n-1} g \circ f^i = 0$. Further, the general solution $\varphi : S \to G$ is given by*

$$\varphi(x) = \sum_{i=0}^{n-1} h\left(f^i(x)\right) + \sum_{i=0}^{n-2} \frac{(n-1-i)}{n} g(f^i(x)), \tag{7}$$

where $h : S \to G$ is an arbitrary function.

Proof. By the recurrent iteration to $\varphi = \varphi \circ f + g$, we have $\sum_{i=0}^{n-1} g \circ f^i = 0$. On the other hand, assume that $\sum_{i=0}^{n-1} g \circ f^i = 0$. Set

$$\varphi(x) := \sum_{i=0}^{n-2} \frac{(n-1-i)}{n} g(f^i(x)) \tag{8}$$

which yields that

$$\begin{aligned}\varphi - \varphi \circ f &= \sum_{i=0}^{n-2} \frac{(n-1-i)}{n} g \circ f^i - \sum_{i=0}^{n-2} \frac{(n-1-i)}{n} g \circ f^{i+1} \\ &= \sum_{i=0}^{n-1} \frac{(n-1-i)}{n} g \circ f^i - \sum_{i=1}^{n-1} \frac{(n-i)}{n} g \circ f^i \\ &= \frac{(n-1)}{n} g - \sum_{i=1}^{n-1} \frac{1}{n} g \circ f^i \\ &= g.\end{aligned}$$

So (8) is a particular solution of $\varphi = \varphi \circ f + g$. By Lemmas 1, 3, (7) is the general solution. □

Theorem 2. *Suppose f is globally periodic with the prime period n on a set S, n is odd, $(G, +)$ is a a uniquely 2-divisible Abelian group. Then $\varphi = -\varphi \circ f + g$ has a unique solution from S to G given by*

$$\varphi(x) = \sum_{i=0}^{n-1} \frac{(-1)^i g(f^i(x))}{2}. \tag{9}$$

Proof. By induction, we have

$$\varphi(f^j(x)) = (-1)^j \varphi(f^j(x)) + \sum_{i=0}^{j-1} (-1)^i g(f^i(x)), \qquad j = 1, 2, \ldots \tag{10}$$

Since n is odd, set $j = n$, then (10) becomes

$$\varphi(x) = -\varphi(x) + \sum_{i=0}^{n-1} (-1)^i g(f^i(x)).$$

Thus (9) follows. One can check that (9) is a particular solution of $\varphi = -\varphi \circ f + g$. By Lemmas 1, 4, (9) is a unique solution. □

Theorem 3. *Suppose f is globally periodic with the prime period n on a set S, n is even, $(G, +)$ is a uniquely n-divisible Abelian group. Then there exists a solution $\varphi : S \to G$ of $\varphi = -\varphi \circ f + g$ if and only if $\sum_{i=0}^{n-1} (-1)^i g(f^i(x)) = 0$. Further, the general solution $\varphi : S \to G$ is given by*

$$\varphi(x) = \sum_{i=0}^{n-1} (-1)^i h\left(f^i(x)\right) + \sum_{i=0}^{n-2} \frac{(-1)^i (n-1-i) g(f^i(x))}{n}, \tag{11}$$

where $h : S \to G$ is an arbitrary function.

Proof. Since n is even, set $j = n$, then (10) becomes

$$\varphi(x) = \varphi(x) + \sum_{i=0}^{n-1} (-1)^i g(f^i(x)),$$

which implies that $\sum_{i=0}^{n-1} (-1)^i g(f^i(x)) = 0$.

On the other hand, assume that $\sum_{i=0}^{n-1} (-1)^i g(\varphi^i(x)) = 0$ holds. Set

$$\varphi(x) := \sum_{i=0}^{n-2} \frac{(-1)^i (n-1-i) g(f^i(x))}{n}. \tag{12}$$

Then one can check that (12) is a particular solution of $\varphi = -\varphi \circ f + g$. By Lemmas 1, 6, (11) is the general solution. □

Remark that the conditions of Theorems 1 and 3 respectively, have a close connection with the following two functional equations

$$\sum_{i=0}^{n-1} \varphi \circ f^i = 0, \qquad n > 2, \tag{13}$$

$$\sum_{i=0}^{n-1} (-1)^i \varphi \circ f^i = 0, \qquad n > 2 \text{ is even}, \tag{14}$$

where f is a given globally periodic map with the prime period n. The general solutions of these two equations are defined with the method of iterative construction in the paper [32]. However, for some applications, it remains interesting to give exact solutions, which are *not* of the form of a piecewise function.

3 Applications and examples

In this section, we conclude with some examples. The interested reader can find exact solutions for more functional equations on the website [35] with a nice classification.

Example 4.1. Find the function $\varphi : (0, +\infty) \to \mathbb{R}$ satisfying $\varphi(x) + \varphi(1/x) = 1$.

Observe that $1/2$ is a particular solution. By Theorem 3, the exact solution of this equation is

$$\varphi(x) = h(x) - h(1/x) + 1/2,$$

where $h : (0, +\infty) \to \mathbb{R}$ is an arbitrary function. With the method of iterative construction in [28, Chp.1] or [32], the general solution with the form of piecewise function is given by

$$\varphi(x) = \begin{cases} \varphi_0(x), & \text{if } x \in (0,1) \\ 1/2, & \text{if } x = 1 \\ 1 - \varphi_0(1/x), & \text{if } x \in (1, \infty) \end{cases}$$

where $\varphi_0 : (0, 1) \to \mathbb{R}$ is an arbitrary function.

Example 4.2. Consider the Knuth mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ in this form [14]

$$T(x, y) = (-y + |x|,\ x),$$

which is globally periodic with the prime period 9.

By Theorem 1, all first integrals of T are of the form $F(x, y) = \sum_{j=0}^{8} h(T^j(x, y))$, where $h : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary non-constant function. In particular, choosing $h(x, y) = y$, we get a first integral

$$F(x, y) = y + |y - |x|| + |x - |y - |x||| + |y - |x - |y||| + |x - |y| + |y - |x - |y||||.$$

Example 4.3. Find the function $\varphi : \mathbb{R} \setminus \{-1, 2\} \to \mathbb{R}$ satisfying $\varphi(x) - \varphi(f(x)) = g(x)$, where $f(x) = \frac{2x-7}{x+1}$ is globally periodic with the prime period 3.

One can examine

$$x \xrightarrow{f} \frac{2x-7}{x+1} \xrightarrow{f} -\frac{x+7}{x-2} \xrightarrow{f} x.$$

By Theorem 1, there exists a solution of this equation if and only if

$$g(x) + g\left(\frac{2x-7}{x+1}\right) + g\left(-\frac{x+7}{x-2}\right) = 0.$$

Further, the exact solution is given by

$$f(x) = \frac{2g(x) + g\left(\frac{2x-7}{x+1}\right)}{3} + h(x) + h\left(\frac{2x-7}{x+1}\right) + h\left(-\frac{x+7}{x-2}\right),$$

where $h : \mathbb{R} \setminus \{-1, 2\} \to \mathbb{R}$ is an arbitrary function.

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References

[1] M. Abate, Discrete holomorphic local dynamical systems, in: G. Gentili, J. Guenot, G. Patrizio eds., *Holomorphic Dynamical Systems*, Lectures notes in Math., Springer Verlag, Berlin, 2010, pp. 1–55.

[2] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

[3] J. Aczél and Z. Daróczy, *On Measures of Information and Their Characterizations, Mathematics in Science and Engineering*, 115, Academic Press, New York, 1975.

[4] J. Aczél and Gy. Maksa, Solution of the rectangular $m \times n$ generalized bisymmetry equation and of the problem of consistent aggregation, *J. Math. Anal. Appl.* 203 (1996), 104–126.

[5] Ch. Babbage, An essay towards a calculus of functions I., *Phil. Trans.* 105 (1815), 389–423.

[6] Ch. Babbage, An essay towards a calculus of functions II., *Phil. Trans.* 106 (1816), 179–256.

[7] K. Baron and W. Jarczyk, Recent results on functional equations in a single variable, perspectives and open problems, *Aequationes Math.* 61 (2001), 1–48.

[8] D. Bell, Associative binary operations and the pythagorean theorem, *The Mathematical Intelligencer* 33 (2011), 92–95.

[9] W. Benz, *Real Geometries*, Bibliographisches Institut, Mannheim, 1994.

[10] M. Bessenyei, Functional equations and finite groups of substitutions, *Amer. Math. Monthly* 117 (2010), 921–927.

[11] M. Bessenyei and Cs.G. Kézi, Functional equations and group substitutions, *Linear Algebra Appl.* 434 (2011), 1525–1531.

[12] M. Bessenyei, G. Horváth and Cs.G. Kézi, *Functional equations on finite groups of substitutions*, *Expo. Math.*, 2012, doi:10.1016/j.exmath.2012.03.004.

[13] V.S. Brodskii and A.K. Slipenko, *Functional Equations*, Visa Skola, Kiev, USSR, 1986.

[14] M. Brown, Problem 6349: A periodic sequence, *Amer. Math. Monthly* 90 (1983), 569. [Solution, ibid, 92 (1985), 218–219.]

[15] E. Castillo and M.R. Ruiz-Cobo, *Functional Equations and Modelling in Science and Engineering*, Marcel Dekker, NewYork, 1992.

[16] E. Castillo, A. Iglesias, and R. Ruiz-Cobo, Functional Equations in Applied Sciences, in: C.K. Chui Ed., *Mathematics in Science and Engineering* 199, Stanford University, Elsevier, 2005.

[17] A.L. Cauchy, Cours d'Analyse de l'Ecole Royale Polytechnique. Chez Debure frères, 1821.

[18] P.-H. Cheung *et al.* ed., *Mathematical Excalibur*, available at http://www.math.ust.hk/excalibur/

[19] A. Cima, A. Gasull and V. Mañosa, Global periodicity and complete integrability of discrete dynamical systems, *J. Difference Equ. Appl.* 12 (2006), 697–716.

[20] I. Daubechies and J.C. Lagarias, Two-scale difference equations I. Existence and global regularity of solutions, *SIAM J. Math. Anal.* 22 (1991), 1388–1410.

[21] I. Daubechies and J.C. Lagarias, Two-scale difference equations II. Local regularity, infinite products of matrices, and fractals, *SIAM J. Math. Anal.* 23 (1992), 1031–1079.

[22] G. Derfel and R. Schilling, Spatially chaotic configurations and functional equations, *J. Phys. A* 29 (1996), 4537–4547.

[23] W. Eichhorn, *Functional Equations in Economics*, Addison-Wesley Educational Publication, 1979.

[24] A. Gasull, V. Mañosa, A Darboux-type theory of integrability for discrete dynamical systems, *J. Difference Equ. Appl.* 8 (2002), 1171–1191.

[25] R. Girgensohn, A survey of results and open problems on the Schilling equation, in: Z. Daróczy and Zs. Páles eds., *Functional Equations – Results and Advance*, Kluwer, 2002, pp. 159–174,

[26] W. Jarczyk, Babbage equation on the circle, *Publ. Math. Debrecen* 63/3 (2003), 389–400.

[27] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Series: Springer Monographs in Mathematics, Springer Dordrecht Heidelberg London, New York, 2008.

[28] M. Kuczma, *Functional Equations in a Single Variable*, PWN-Polish Scientific Publishers, Warsaw, 1968.

[29] M. Kuczma, B. Choczewski and R. Ger, *Iterative Functional Equations*, Cambridge University Press, Cambridge, 1990.

[30] K. Lajkó, *Functional Equations in Competition Problems*, University Press of Debrecen, Debrecen, Hungary, 2005.

[31] Z. Leśniak and Y.-G. Shi, One class of planar rational involutions, *Nonlinear Anal.* 74 (2011), 6097–6104.

[32] A. Mach, On some functional equations involving Babbage equation, *Results. Math.* 51 (2007) 97–106.

[33] L. Molnár, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces, Lecture Notes in Mathematics*, 1895, Springer-Verlag, Berlin, 2007.

[34] Gy. Nagy, ed., *KöMaL – Mathematical and Physical Journal for Secondary Schools*, available at http://www.komal.hu

[35] A.D. Polyanin and A.V. Manzhirov, *Handbook of Integral Equations: Exact Solutions* (Supplement. Some Functional Equations) [in Russian], Faktorial, Moscow, 1998, available at http://eqworld.ipmnet.ru

[36] S. Presić, Méthode de résolution d'une classe d'équations fonctionnelles linéaires, Univ. Beograd, Publ. Elektrotechn. Fak. Scr. Math. Fiz 119 (1963), 21–28.

[37] S. Presić, Sur l'équation fonctionnelle $f(x) = H(x, f(x), f(\theta_1 x), \ldots, f(\theta_n x))$, Univ. Beograd, Publ. Elektrotechn. Fak. Scr. Math. Fiz 118 (1963), 17–20.

[38] J.F. Ritt, On certain real solutions of Babbage's functional equation, *Ann. Math.* 17 (1916), 113–122.

[39] E. Schröder, Ueber iterirte Functionen, *Math. Ann.* 3 (1879), 296–322.

[40] Y.-G. Shi and L. Chen, Meromorphic iterative roots of linear fractional functions, *Sci. China Ser. A* 52 (2009), 941–948.

[41] C.G. Small, *Functional Equations and How to Solve Them*, Springer Science+Buisiness Media, LLC, New York, USA, 2007.

[42] L. Székelyhidi, Discrete Spectral Synthesis and Its Applications, Springer Monographs in Mathematics, Springer-Verlag, Dordrecht, 2006.

[43] J.-C. Yoccoz, Analytic linearization of circle diffeomorphisms, in: Cetraro, ed., *Dynamical Systems and Small Divisors*, 1998, Lecture Notes in Math., 1784, Springer-Verlag, New York, 2002, pp. 125–173.

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