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On Beckner's inequality for Gaussian measures

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The standard Gaussian measure on euclidean space $\mathbb{R}^n$,

$$\gamma(dx) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}\, dx,$$

has many fascinating properties, among them the Poincaré inequality

$$\|f\|_2 \le \|\nabla f\|_2 \quad \text{for} \quad \int_{\mathbb{R}^n} f\, d\gamma = 0$$

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Das Gaußsche Maß auf $\mathbb{R}^n$ besitzt zahlreiche schöne Eigenschaften. Einige davon tauchen im Zusammenhang mit verschiedenen Normen bei Ungleichungen auf. Die Poincaré-Ungleichung und die logarithmische Sobolev-Ungleichung von Gross sind zwei prominente Beispiele. 1989 bewies Beckner eine L^p-Ungleichung für $1 \le p < 2$, welche zwischen den beiden genannten Ungleichungen interpoliert: Die Poincaré-Ungleichung erhält man für $p = 1$, die Ungleichung von Gross für $p \to 2$. Die Autoren der vorliegenden Arbeit benutzen nun die Tatsache, dass das Gaußsche Maß als Wärmeleitungskern auftritt, um mit Hilfe der klassischen Wärmeleitungshalbgruppe Beckners Ungleichung neu zu beweisen und sie gleichzeitig auf den Fall $p > 2$ auszudehnen.

and Gross's [6] logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} f^2 \log|f|\, d\gamma - \|f\|_2^2 \log\|f\|_2 \le \|\nabla f\|_2^2.$$

Beckner [4] has proved the functional inequality

$$\|f\|_2^2 - \|f\|_p^2 \le (2-p)\|\nabla f\|_2^2, \quad 1 \le p < 2. \tag{1}$$

For $p = 1$, inequality (1) is equivalent to the Poincaré inequality, as can be seen for bounded f by adding a sufficiently large constant C so that $f + C$ is nonnegative, and for a general f by approximation by bounded functions. Furthermore, if we divide both sides of (1) by $2-p$ and let $p \to 2$, the left side tends to the left side of the logarithmic Sobolev inequality. Thus Beckner's inequality interpolates between the Poincaré inequality and the logarithmic Sobolev inequality.

Beckner's original proof of (1) is based on the explicit spectral decomposition of the Ornstein–Uhlenbeck operator in terms of Hermite polynomials and Nelson's [9] hypercontractivity inequality for the Ornstein–Uhenbeck semigroup. Apparently unaware of Beckner's work at the time, Latała and Oleszkiewicz [7] proved an extension of Beckner's inequality for measures $ce^{-|x_1|^r - \cdots - |x_n|^r}dx$ with $1 \le r \le 2$. However, in the Gaussian case $r = 2$ the inequality (1) was derived from the logarithmic Sobolev inequality and the hypercontractivity of the Ornstein–Uhlenbeck semigroup, via an argument similar to that in Beckner [4]. Many other authors also studied Beckner's inequality and its generalizations in various directions; see, e.g., Arnold, Bartier, and Dolbeault [1]; Arnold, Markowich, Toscani, and Unterreiter [2]; Barthe and Roberto [3]; Chafai [5]; Ledoux [8]; and Wang [11]. But none of these works includes a proof of (1) which does not rely on ideas or results comparable in difficulty to the logarithmic Sobolev inequality or its consequence the hypercontractivity. In addition, most of these works prove Beckner's inequality in a much broader setting than that in which Beckner originally derived it, which can make it difficult for a reader without susbstantial background in the field to discern the beauty and simplicity of the original inequality. This situation makes it desirable and instructive to search for a more direct proof of Beckner's inequality. In this note, we shall demonstrate this possibility by proving the following slight extension of Beckner's inequality by an elementary argument based on the classical heat semigroup.

Theorem. *Let $q \ge 2$ and $1 \le p \le q$. Then if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function such that f and each of its partial derivatives belong to $L^q(\mathbb{R}^n)$, we have*

$$\|f\|_q^2 - \|f\|_p^2 \le (q-p)\|\nabla f\|_q^2. \tag{2}$$

Remark 1. We state the inequality here for smooth functions for expository purposes, but an elementary approximation argument shows that it is also valid for functions f in the Sobolev space $W^{1,q}(\mathbb{R}^n)$.

The basic tool for our proof is the classical heat semigroup $\{P_s\}$ defined by

$$P_s f(x) = \frac{1}{(2\pi s)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/2s}\, dy.$$

Note that if f is bounded and continuous, then $P_s f \to f$ as $s \to 0$, and if $f \in L^1(\gamma)$, then

$$P_1 f(0) = \int_{\mathbb{R}^n} f \, d\gamma.$$

Furthermore, it is easy to verify from the definition that the heat semigroup has the following properties:

$$P_s P_t = P_{s+t}, \quad \partial_s P_s = \frac{1}{2} \Delta P_s = \frac{1}{2} P_s \Delta, \quad \nabla P_s = P_s \nabla.$$

Here ∇ and Δ are the usual gradient and Laplace operator on $\mathbb{R}^n$, respectively. Aside from these elementary properties, the only other tool we will need for the proof of our main result (2) is Hölder's inequality for a Borel measure ν on $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} fg \, d\nu \le \left(\int_{\mathbb{R}^n} |f|^p \, d\nu \right)^{1/p} \left(\int_{\mathbb{R}^n} |g|^q \, d\nu \right)^{1/q} \tag{3}$$

for $f \in L^p(\mathbb{R}^n, \nu)$, $g \in L^q(\mathbb{R}^n, \nu)$, and exponents $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. By replacing f with $|f|$ and then approximating $|f|$ by smooth positive functions bounded away from 0 and ∞, it is enough to show the inequality (2) for a smooth function f such that $0 < c \le f \le C$. For $0 \le s \le 1$, consider the function

$$\phi_s(x) = \left[P_s \left(P_{1-s} f^p \right)^{q/p} (x) \right]^{2/q}. \tag{4}$$

We can write the left side of (2) as

$$\|f\|_q^2 - \|f\|_p^2 = \phi_1(0) - \phi_0(0) = \int_0^1 \partial_s \phi_s(0) ds.$$

The idea of considering such a function in the context of functional inequalities can be traced back to Neveu [10].

The technical part of our proof is a straightforward computation of the derivative of (4) with respect to s, which will lead to a convenient expression for this derivative (see (7) below). From this, we will repeatedly apply Hölder's inequality to get the simple upper bound

$$\partial_s \phi_s(0) \le (q - p) \left(\int_{\mathbb{R}^n} |\nabla f|^q d\gamma \right)^{2/q},$$

from which our desired inequality (2) follows immediately by integrating with respect to s from 0 to 1.

From the definition (4) of ϕ_s we have

$$\partial_s \phi_s = \partial_s \left[P_s g_s^{q/p} \right]^{2/q} = \frac{2}{q} a_s \partial_s (P_s g_s)^{q/p},$$

where, to simplify the notation hereafter, we have introduced the functions

$$g_s = P_{1-s} f^p \quad \text{and} \quad a_s = \left(P_s g_s^{q/p} \right)^{2/q - 1}.$$

The derivative $\partial_s (P_s g_s)^{q/p}$ can be easily calculated and we obtain

$$\partial_s \phi_s = \frac{2}{q} a_s (\partial_s P_s) g_s^{q/p} + \frac{2}{p} a_s P_s \left(g_s^{q/p-1} \partial_s g_s \right). \tag{5}$$

Using the relation $\partial_s P_s = (1/2) P_s \Delta$, we may rewrite the first term on the right side as $(1/q) a_s P_s \Delta \left(g_s^{q/p} \right)$, which equals

$$\frac{1}{p} \left(\frac{q}{p} - 1 \right) a_s P_s \left(g_s^{q/p-2} |\nabla g_s|^2 \right) + \frac{1}{p} a_s P_s \left(g_s^{q/p-1} \Delta g_s \right) \tag{6}$$

by the identity

$$\Delta \left(h^{q/p} \right) = \frac{q}{p} \left(\frac{q}{p} - 1 \right) h^{q/p-2} |\nabla h|^2 + \frac{q}{p} h^{q/p-1} \Delta h$$

applied with $h = g_s$. From $\partial_s P_{1-s} = -(1/2) \Delta P_{1-s}$ we have $\partial_s g_s = -(1/2) \Delta g_s$, so the second term in the sum (6) exactly cancels the second term in (5). In the remaining term, we use the fact that P_{1-s} commutes with ∇ to write $\nabla g_s = p P_{1-s} (f^{p-1} \nabla f)$. This gives

$$\partial_s \phi_s = (q - p) a_s P_s \left(g_s^{q/p-2} |P_{1-s} (f^{p-1} \nabla f)|^2 \right). \tag{7}$$

Note that P_{1-s} is an integral with respect to a (probability) measure, so we can use Hölder's inequality (3) with the exponents $p/(p-1)$ and p to get

$$|P_{1-s}(f^{p-1} \nabla f)| \le P_{1-s}(f^{p-1} |\nabla f|) \le \left(P_{1-s} f^p \right)^{(p-1)/p} \left(P_{1-s} |\nabla f|^p \right)^{1/p}.$$

Thus, by (7),

$$\partial_s \phi_s \le (q - p) a_s P_s \left(g_s^{q/p-2/p} \left(P_{1-s} |\nabla f|^p \right)^{2/p} \right). \tag{8}$$

The case $q = 2$ is covered by trivial modifications to what follows, so in the remainder of the proof we assume $q > 2$. A second application of Hölder's inequality with the exponents $q/(q-2)$ and $q/2$ yields

$$P_s \left(g_s^{q/p-2/p} \left(P_{1-s} |\nabla f|^p \right)^{2/p} \right) \le \left(P_s g_s^{q/p} \right)^{1-2/q} \left(P_s \left(P_{1-s} |\nabla f|^p \right)^{q/p} \right)^{2/q}.$$

The first factor on the right side is exactly a_s^{-1}, which cancels the factor a_s in (8). We thus have

$$\partial_s \phi_s \le (q - p) \left(P_s \left(P_{1-s} |\nabla f|^p \right)^{q/p} \right)^{2/q}. \tag{9}$$

Since $1 \le p \le q$, another application of Hölder's inequality gives

$$P_{1-s} |\nabla f|^p \le \left(P_{1-s} |\nabla f|^q \right)^{p/q}.$$

This together with the semigroup property $P_s P_{1-s} = P_1$ gives

$$\left(P_s \left(P_{1-s} |\nabla f|^p \right)^{q/p} \right)^{2/q} \le \left(P_s P_{1-s} |\nabla f|^q \right)^{2/q} = \left(\int_{\mathbb{R}^n} |\nabla f|^q \, d\gamma \right)^{2/q}.$$

The last equality holds after evaluating at $x = 0$. It follows from (9) that

$$\partial_s \phi_s(0) \le (q-p) \left(\int_{\mathbb{R}^n} |\nabla f|^q d\gamma \right)^{2/q} .$$

Integrating from $s = 0$ to $s = 1$ yields the desired inequality (2).

We conclude this note with a few more remarks.

Remark 2. The constant $q - p$ on the right side of our new inequality (2) cannot be improved. This can be seen by taking $f(x) = e^{tx_1}$ for $t > 0$, calculating both sides explicitly, and letting $t \to 0$.

Remark 3. The condition $q \ge 2$ in (2) is essential. Indeed, an inequality of the form

$$\|f\|_q^2 - \|f\|_p^2 \le C(q-p)\|\nabla f\|_q^2 \tag{10}$$

cannot hold in the parameter range $1 \le p < q < 2$ with any constant C. Replacing f by $1 + \epsilon f$ in (10) and comparing the coefficients of ϵ^2 in the Taylor expansions of both sides, we see that (10) would lead to

$$\|f\|_2^2 - \left(\int_{\mathbb{R}^n} f \, d\mu \right)^2 \le \|\nabla f\|_q^2.$$

Taking again $f(x) = e^{tx_1}$, this time with a very large t, we see easily that this inequality cannot hold if $q < 2$.

Remark 4. However, the function

$$\theta(q, p) = \frac{\|f\|_q^2 - \|f\|_p^2}{1/p - 1/q}$$

is increasing in both arguments whenever $1 \le p < q$ (see Latała and Oleszkiewicz [7]). This fact together with the original Beckner's inequality (1) implies

$$\|f\|_q^2 - \|f\|_p^2 \le \frac{2}{q}(q-p)\|\nabla f\|_2^2, \quad \text{for } 1 \le p \le q \le 2.$$

Remark 5. In Section 3.1 of [8], Ledoux used a nonlinear partial differential equation to prove a version of (1) for the invariant probability measures of a Markov semigroup whose generator satisfies a curvature-dimension inequality. In the Gaussian case, his inequality reduces to a sharpened form of (1), with the right side multiplied by $(n-1)/n$ and the parameter p allowed to increase to $2n/(n-1)$.

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