Elemente der Mathematik

A characterization of the ellipse related to illumination bodies

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1 Introduction

Let $\mathcal{E}$ be an ellipse in the plane with center at the origin and let $\mathcal{E}'$ be a homothetic copy of $\mathcal{E}$ with center of homothety at the origin and ratio of homothety bigger than 1. It is not difficult to see the veracity of the following facts.

(1) The midpoints of every set of parallel chords of $\mathcal{E}$ are collinear. They are indeed over the conjugate diameter to the direction of the chords.

(2) The midpoint of every chord of $\mathcal{E}'$ which is tangent to $\mathcal{E}$, belongs to $\mathcal{E}$, and the chord cuts off from conv$(\mathcal{E}')$ a set with constant area.

(3) The area of conv$(p \cup \mathcal{E})$ is constant for every $p \in \mathcal{E}'$ and the segment joining the two contact points, between the tangents to $\mathcal{E}$ from p with $\mathcal{E}$, is parallel to the tangent line to $\mathcal{E}'$ at p.

We wonder if there is another convex body, besides ellipses, which has at least one of the properties explained in (1), (2) or (3). By a theorem proved by W. Blaschke [1] and related

Die Mittelpunkte paralleler Sehnen einer Ellipse liegen bekanntlich auf einer Geraden (nämlich auf dem zur Richtung der Sehnen konjugierten Durchmesser). Nach einem Satz von Karl Hermann Brunn charakterisiert diese Eigenschaft die Ellipsen in der Klasse der beschränkten konvexen Kurven. Eine Ellipse E hat noch zahlreiche weitere schöne Eigenschaften, etwa diese: Die Menge der Punkte P ausserhalb von E mit der Eigenschaft, dass der Flächeninhalt der konvexen Hülle von $\{P\} \cup E$ konstant ist, ist eine zu E ähnliche Ellipse. Charakterisiert auch diese Eigenschaft die Ellipsen? Lesen Sie weiter!

by him to Brunn, we know the answer for the property (1): *if K is a planar convex body such that the midpoints of every set of parallel chords are aligned then it is an ellipse.*

With respect to properties (2) and (3) there are some results, but we first need to introduce the definition of two important convex bodies. Given a convex body K, i.e., a compact convex set in $\mathbb{R}^n$ with non-empty interior, the floating and illumination bodies of K are two classes of very important convex bodies associated to K. They are defined as follows: for a given positive number $\delta < \mathrm{vol}_n(K)$, the *floating body* denoted as K_δ is defined as the intersection of all the closed half-spaces which cut off from K a cap with volume δ. For instance, if K is a Euclidean disc with unit area in the plane and $\delta < 1/2$, then K_δ is a disc concentric with K. Notice that the midpoint of every chord of K which is tangent to K_δ belongs to K_δ. This property holds for every convex body and its floating body K_δ (see for instance [4]).

For a positive number δ, *the illumination body K^δ* of K is defined as the set

$$K^\delta = \{x \in \mathbb{R}^n : \mathrm{vol}_n(\mathrm{conv}(x \cup K)) - \mathrm{vol}_n(K) \le \delta\}.$$
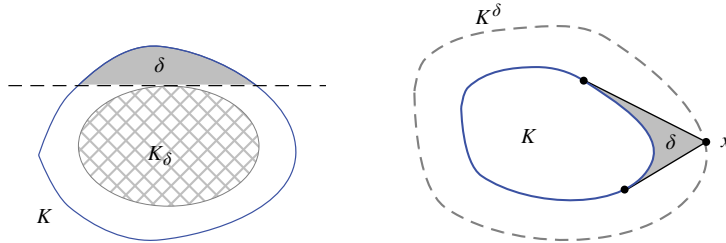

Figure 1 The floating and illumination bodies

For instance, if K is a square of unit area and $\delta = \frac{\sqrt{2}}{4}$, it is easy to see that its illumination body K^δ is an octagon of side length 1, as shown in Figure 2. Notice that for every line ℓ supporting K^δ at a point x, there is a chord of K joining two contact points with the tangent lines from x, which is parallel to ℓ. This property holds for every convex body K and its illumination body K^δ as will be shown in Lemma 1 in the next section. In some sense this property is the counterpart for illumination bodies of the property that the midpoint of any chord of K tangent to K_δ is in K_δ.
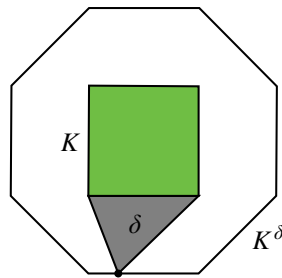

Figure 2 The illumination body of a square

One interesting and important problem in Convex Geometry related to the floating body is the homothety conjecture raised by C. Schütt and E. Werner in [5]: *Does a convex body K have to be an ellipsoid, if K is homothetic to K_δ for some $\delta > 0$?*

In [5] indeed, they proved that if there is a sequence $\delta_k \longrightarrow 0$ such that K_{δ_k} is homothetic to K for all $k \in \mathbb{N}$ (with respect to the same center of homothety), then K is an ellipsoid. Later, A. Stancu [7] gave a proof of the conjecture under the consideration that the boundary of K is of class $C^{\geq 4}$. Finally, in [10], E. Werner and D. Ye proved the conjecture. With respect to the illumination body, A. Stancu proved in [8] the following: *Let $K \subset \mathbb{R}^{n+1}$ be a convex body of class C_+^2. There exists a positive number $\delta(K)$ such that K^δ is homothetic to K with respect to the same center of homothety, for some $\delta < \delta(K)$ if and only if K is an ellipsoid.* However, it seems that Stancu's theorem does not deal with the planar case.

Here we are interested in the illumination body problem for the planar case, that is, we will prove the following.

Theorem 1. *Let $K \subset \mathbb{R}^2$ be a strictly convex body with boundary of class C^1 such that for all $\delta > 0$ the illumination body K^δ is homothetic to K with respect to a point $x \in \text{int } K$. Then K is an ellipse.*

2   A characterization of the ellipse

Let Γ and γ be two closed convex curves in the plane with $\gamma \subset \text{int}(\text{conv } \Gamma)$. We will say that a pair of curves with this condition is a *nested pair* of curves. For every point $x \in \Gamma$ let $a, b \in \gamma$ be the first two contact points of the left and right tangents through x, respectively, with γ. Denote the area of the region enclosed between the segments $[a, x]$, $[b, x]$, and the arc $\widehat{ab}$ by $A(x)$ (see Figure 3).
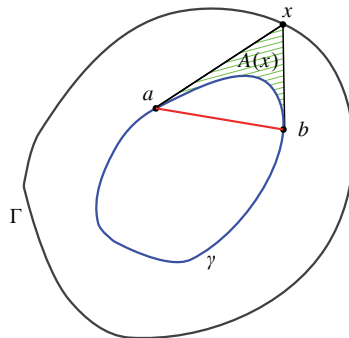

Figure 3 The region $A(x)$

Using this notation we have that $A(x)$ is constant if and only if Γ is the boundary of an illumination body of conv γ. As a first result we prove the following lemma which is necessary for the proof of Theorem 1.

Lemma 1. *Let Γ and γ be a nested pair of curves. Suppose conv Γ is an illumination body of conv γ and let x be any point in Γ. Then, for every chord $[a, b] \subset \text{conv } \gamma$ joining two contact points with the tangents to γ from x, the line ℓ parallel to $[a, b]$ through x is a support line of Γ.*
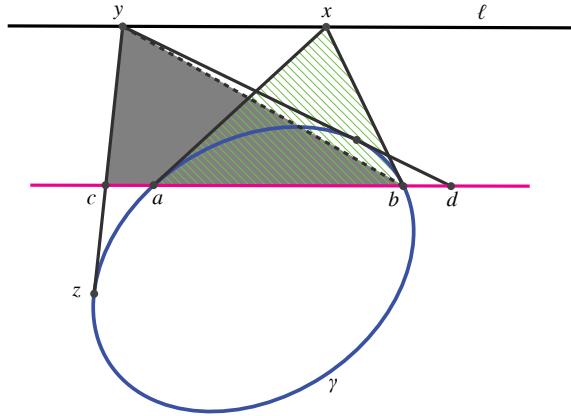
Figure 4 The line supporting Γ at x is parallel to $[a, b]$

Proof. Let x be any point in Γ and let a and b be two contact points of γ with the tangent lines from x, as shown in Figure 4. Suppose ℓ is a line through x which is parallel to the chord $[a, b]$. Since γ is contained in the region bounded by the rays $\overrightarrow{xa}$ and $\overrightarrow{xb}$ we have that ℓ does not intersect γ. Let $y \in \ell$ be a point different from x and let c and d be the points where the two supporting lines of γ intersect the line ab. By the convexity of γ we have that $[a, b] \subset [c, d]$. Assume that y is to the left of x, then c is contained in the segment $[y, z]$ with z the first contact point of the tangent line from y (it could be that z coincides with c). It follows that

$$|xab| = |yab| \leq |ycb| \leq |\text{conv}(y \cup \gamma)|.$$

The case when y is to the right of x is analogous, therefore, we have that ℓ is a supporting line of Γ.   □

This lemma directly implies the following.

Corollary 1. *If $K \subset \mathbb{R}^2$ is strictly convex then ∂K^δ is of class C^1.*

Given a convex body K in the plane, we say that a chord $[a, b]$ is an affine diameter if there is a pair of parallel lines supporting K at a and b, respectively. For completeness we will prove the following known lemmas (see for instance [2], and [3]).

Lemma 2. *Let $K \subset \mathbb{R}^2$ be a strictly convex body with boundary of class C^1, and let x be a point in* int K *such that every chord of K through x is an affine diameter. Then K has center of symmetry at x.*

Proof. WLOG we may suppose that x coincides with the origin 0. Suppose the boundary of K is parameterized in polar coordinates by $(\theta, r(\theta))$, with $\theta \in [0, 2\pi]$. Let $\ell(\theta)$ and $\ell(\theta + \pi)$ be the corresponding supporting lines, as shown in Figure 5, and denote by $\beta(\theta)$ the directed angle between the radius vector and the tangent line at the point $(\theta, r(\theta))$, for
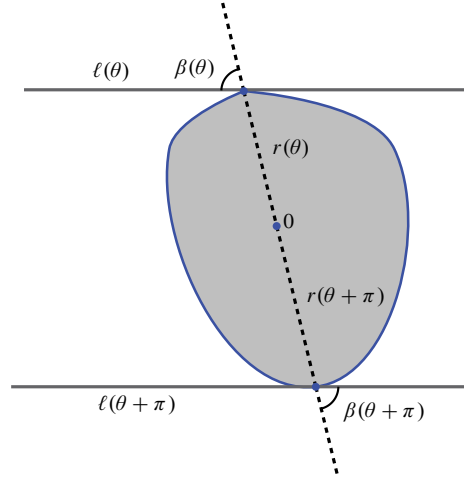
Figure 5 K has center of symmetry at 0

every $\theta \in [0, 2\pi]$. By the known formula (see for instance [9])

$$\tan(\beta(\theta)) = \frac{r(\theta)}{r'(\theta)}$$

and since $\beta(\theta) = \beta(\theta + \pi)$ we have that

$$\frac{r(\theta)}{r'(\theta)} = \frac{r(\theta + \pi)}{r'(\theta + \pi)},$$

which implies that

$$\frac{d}{d\theta}\left(\frac{r(\theta)}{r(\theta + \pi)}\right) = \frac{r'(\theta) \cdot r(\theta + \pi) - r(\theta) \cdot r'(\theta + \pi)}{[r(\theta + \pi)]^2} = 0.$$

Hence, $r(\theta) = \lambda \cdot r(\theta + \pi)$, for some number λ and for every $\theta \in [0, 2\pi]$. Now, by the Intermediate Value Theorem, it is easy to see that λ must be equal to 1. Therefore, K has center of symmetry at 0. $\qquad\square$

Lemma 3. *Let $K \subset \mathbb{R}^2$ be a strictly convex body with boundary of class C^1, and let ℓ be a line intersecting* int K *with the following property: for any point $p \in \{\ell \setminus K\}$, the line joining the two contact points a_p, b_p, of the tangents drawn from p with K, is parallel to a fixed vector $v \in \mathbb{S}^1$ and intersects ℓ in a point x_p, such that a_p, x_p, b_p are in that order. Then the ratio*

$$\frac{|a_p x_p|}{|b_p x_p|}$$

is constant for all $p \in \{\ell \setminus K\}$.

Proof. First we apply an affine transformation T in the plane in such a way that ℓ is orthogonal to $[a_p, b_p]$ for every $p \in \{\ell \setminus K\}$. WLOG we may assume that $\ell' = T(\ell)$

coincides with the x-axis. Set $a'_p = T(a_p)$, $b'_p = T(b_p)$, and $K' = T(K)$, and let γ_1 be the part of bd K' contained in the upper half-plane, and let γ_2 be the part of bd K' in the lower half-plane. Let 0 and d be the two points where ℓ' intersects bd K'. We may suppose that γ_1 and γ_2 coincide with the two differentiable functions $f, g : [0, d] \longrightarrow \mathbb{R}$ in the interval $[0, d]$. Since tangentiality and concurrency are preserved under the application of an affine transformation we have that the lines tangent to K' at the points $a'_p = (x_0, f(x_0))$ and $b'_p = (x_0, g(x_0))$ concur at the point $T(p)$ on ℓ'. By a simple calculation we have

$$\frac{f'(x_0)}{f(x_0)} = \frac{g'(x_0)}{g(x_0)}.$$

It follows that $f'(x_0) \cdot g(x_0) - g'(x_0) \cdot f(x_0) = 0$, and since $g(x_0) \neq 0$ we have that

$$\frac{f'(x_0) \cdot g(x_0) - g'(x_0) \cdot f(x_0)}{[g(x_0)]^2} = 0.$$

Hence

$$\frac{d}{dx}\left(\frac{f(x_0)}{g(x_0)}\right) = 0.$$

Now, since x_0 is an arbitrary point in the interval $(0, d)$ we have that there is a real number k such that $\frac{f(x)}{g(x)} = k$, for every $x \in (0, d)$. Therefore, $\frac{|a'_p x'_p|}{|b'_p x'_p|} = |k|$ and so $\frac{|a_p x_p|}{|b_p x_p|} = |k|$. $\square$
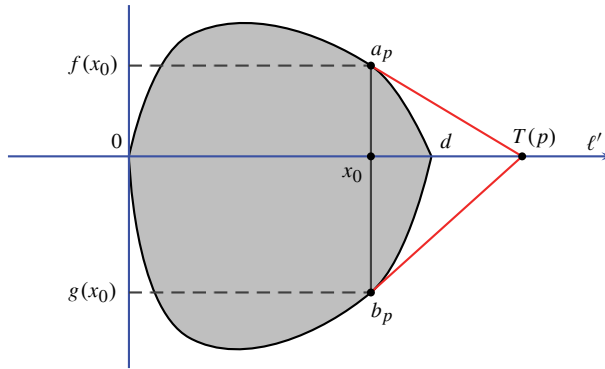

Figure 6 The ratio $\frac{|a_p x_p|}{|b_p x_p|}$ is constant

As an interesting application of Lemma 1 we prove the characterization of the ellipse mentioned in the introduction.

Theorem 1. *Let $K \subset \mathbb{R}^2$ be a strictly convex body with boundary of class C^1 such that for all $\delta > 0$ the illumination body K^δ is homothetic to K with respect to a point $x \in$ int K. Then K is an ellipse.*

Proof. WLOG we may assume that x coincides with the origin 0.

Claim 1. K is centrally symmetric with center of symmetry at 0.

Proof. Let $v \in \mathbb{S}^1$ be a given direction and $[a, b]$ a chord of K parallel to v. Suppose the supporting lines of K at a and b intersect at a point p (see Figure 7). For some $\delta \in \mathbb{R}^+$ we have $p \in \partial K^\delta$. By Lemma 1 we have that the supporting line of K^δ through p is parallel to $[a, b]$. Let q be the point of intersection between ∂K and the ray $\overrightarrow{0p}$. Since K and K^δ are homothetic with center of homothety at 0 we have that the support line of K through q is also parallel to $[a, b]$. Let $[c, d]$ be the chord of K with maximum length in direction v. It is known that there exists a pair of parallel supporting lines of K through c and d (see for instance [6]). Let $\omega \in \mathbb{S}^1$ be the unit vector parallel to these supporting lines and denote by $\infty(\omega)$ the point at infinity in direction ω.
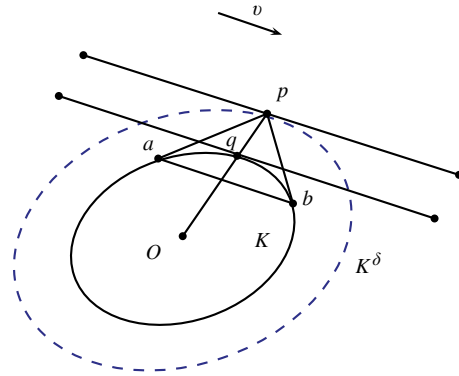

Figure 7

Consider a sequence of chords $\{\ell_n\}$ parallel to $[a, b]$, such that the limit of the sequence is the chord $[c, d]$ and such that they are contained in the same half-plane delimited by $[c, d]$. For a given $n \in \mathbb{N}$ let p_n be the intersection point of the supporting lines of K through the extreme points of ℓ_n. We have that 0, q, and p_n are aligned. On the other hand, we have that $p_n \to \infty(\omega)$ when $n \to \infty$. It follows that the ray $\overrightarrow{0q}$ is parallel to ω. Now, let t be the point where the other supporting line of K parallel to v intersects ∂K. Analogously, we prove that $\overrightarrow{0t}$ is parallel to ω and 0, q, and t are collinear. Since through q and t there exist parallel supporting lines, we have that $[q, t]$ is an affine diameter of K through 0. The direction v was chosen arbitrarily and since K is strictly convex, then it is easy to see that all the affine diameters of K meet at 0. By Lemma 2 we have that K is centrally symmetric with center of symmetry at 0. $\qquad\square$

Claim 2. The midpoints of every set of parallel chords are collinear.

Proof. Consider an arbitrary direction $v \in \mathbb{S}^1$. Let ℓ_1 and ℓ_2 be the two supporting lines of K with direction v and let $\{s\} = \ell_1 \cap \partial K$ and $\{t\} = \ell_2 \cap \partial K$. Given any chord parallel to v, the supporting lines through its extreme points intersect on the line st. By Lemma 3 we have that any chord parallel to v is divided by the segment $[s, t]$ in a constant ratio. The segment $[s, t]$ contains 0 in its interior, so the chord of K through 0 and parallel to v is divided by 0 in the ratio $1 : 1$. We have shown that the midpoints of the chords parallel to v are aligned. $\qquad\square$

Finally, we apply the theorem of Blaschke mentioned in the introduction and the proof is complete.   □

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