Elem. Math. 70 (2015) 89 – 94
0013-6018/15/030089-6
DOI 10.4171/EM/281

Elemente der Mathematik

On the convergence of thinned harmonic series

Wolfgang Stadje

Wolfgang Stadje received his doctoral degree and his habilitation in mathematics at the University of Göttingen in 1978 and 1981, respectively. He is a professor at the University of Osnabrück. His main areas of research are applied probability and stochastic processes in operations research.

In 1914 Kempner [5] showed that the series of all reciprocals of natural numbers without the digit 9 in their decimal expansion converges. This series turned out to be extremely slowly convergent to 22.92067… (the first 100 digits of the limit were computed in [3]); for example the sum of its first 10^{28} terms is still less than 22 [2]. In 1916 Irwin [4] proved that any harmonic series restricted to integers which contain some digits exactly certain prescribed numbers of times converges, too. Techniques for the numerical calculation of these "Kempner–Irwin series" were presented in [2]. More results on series of this type can be found in [1], [7], [8] and the references therein.

In this note we study Kempner–Irwin series with the reciprocals n^{-1} replaced by the powers $n^{-\alpha}$, $\alpha > 0$. It is shown that there is an $\alpha_0 \in [0, 1)$ such that the series diverges if and only if $\alpha \leq \alpha_0$. If we consider d-ary expansions (d being an integer larger than one) and $0 \leq k < d$ digits are restricted, then

$$\alpha_0 = \log_d(d - k).$$

Die Divergenz der harmonischen Reihe ist eine der Grundtatsachen der Analysis. Genauer gilt bekanntlich, dass die Reihe $\sum n^{-\alpha}$ nur für $\alpha > 1$ konvergiert. Vor etwa hundert Jahren hat Kempner gezeigt, dass aus der harmonischen Reihe durch Weglassen aller Kehrwerte von Zahlen mit einer Neun in der Dezimaldarstellung eine konvergente Reihe entsteht, und kurz danach hat Irwin dieses Ergebnis auf die Reihen erweitert, die sich ergeben, wenn man nur über solche natürlichen Zahlen summiert, in deren Dezimaldarstellung einige der Ziffern jeweils in vorgegebener Anzahl (oder gar nicht) auftreten. In diesem Artikel wird unter anderem gezeigt, dass es für diese Kempner–Irwin-Reihen einen Schwellenwert $\alpha_0 \in [0, 1)$ gibt, so dass die entsprechende Reihe $\sum n^{-\alpha}$ genau für $\alpha > \alpha_0$ konvergiert: Es gilt $\alpha_0 = \log_d(d - k)$, wenn man d-adische Entwicklungen betrachtet und über das Auftreten von k Ziffern Vorschriften macht.

Here $\log_d$ denotes the logarithm with respect to the base d. Note that in terms of natural logarithms the threshold value can be expressed as $\alpha_0 = \ln(d-k)/\ln d$. It is interesting that α_0 only depends on the base d and the number k of digits involved in the constraints.

Fix $k \in \{1, \ldots, d-1\}$, digits $a_1, \ldots, a_k \in \{1, \ldots, d-1\}$ and integers $n_1, \ldots, n_k \geq 0$. Setting $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{n} = (n_1, \ldots, n_k)$, let $A(\mathbf{a}, \mathbf{n})$ be the set of all positive integers that contain the digit a_i exactly n_i times in their d-ary expansion for $i = 1, \ldots, k$. Finally let, for $\alpha > 0$,

$$S(\alpha, \mathbf{a}, \mathbf{n}) = \sum_{n \in A(\mathbf{a}, \mathbf{n})} n^{-\alpha}.$$

The series $S(\alpha, \mathbf{a}, \mathbf{n})$ obviously converges for $\alpha > 1$ and, by Irwin's result, also for $\alpha = 1$. What happens for $\alpha \in (0, 1)$? We prove

Theorem 1 *$S(\alpha, \mathbf{a}, \mathbf{n})$ is divergent if and only if $\alpha \leq \log_d(d-k)$.*

In the case of convergence we also derive an upper bound. For this we need the sum of the one-digit terms of $S(\alpha, \mathbf{a}, \mathbf{n})$:

$$s(\alpha, \mathbf{a}, \mathbf{n}) = \sum_{n \in A(\mathbf{a}, \mathbf{n}) \cap \{1, \ldots, d-1\}} n^{-\alpha}.$$

This sum is zero if some n_i is larger than 1. If exactly one n_i is equal to one, say $n_{i_0} = 1$, and the other n_i's are zero, then $s(\alpha, \mathbf{a}, \mathbf{n}) = a_{i_0}^{-\alpha}$. If all n_i's are zero, then

$$s(\alpha, \mathbf{a}, \mathbf{n}) = \sum_{i \in \{1, \ldots, d-1\} \setminus \{a_1, \ldots, a_k\}} n^{-\alpha} = \sum_{n=1}^{d-1} n^{-\alpha} - \sum_{p=1}^{k} a_p^{-\alpha}.$$

Theorem 2 *If $\alpha > \log_d(d-k)$, then we have, setting $N = \sum_{p=1}^{k} n_p$,*

$$\begin{aligned} S(\alpha, \mathbf{a}, \mathbf{n}) \leq s(\alpha, \mathbf{a}, \mathbf{n}) + \frac{d^\alpha}{(k + d^\alpha - d)^N n_1! \cdots n_k!} \\ \times \left[\frac{N!}{k + d^\alpha - d} \left(\sum_{n=1}^{d-1} n^{-\alpha} - \sum_{p=1}^{k} a_p^{-\alpha} \right) + (N-1)! \sum_{p=1}^{k} n_p a_p^{-\alpha} \right]. \end{aligned} \tag{1}$$

1 Examples

1. For $k = d - 1$ the frequency of all digits $1, \ldots, d-1$ is prescribed, so that only the number of zeros and the ordering of the constrained digits are free. In this case Theorem 1 states that the resulting series is convergent for all $\alpha > 0$.

2. For $k = d - 2$ the frequency of all but one of the digits $1, \ldots, d-1$ is prescribed, so that only the number of zeros and that of one other digit are free. Then Theorem 1 states that any resulting series of this kind is divergent for all $\alpha \leq \log_d 2 = \ln 2/\ln d$. For example, for $d = 10$ (decimal expansions) and any series in which the numbers of the digits $2, 3, \ldots, 9$ are prescribed, convergence takes place if and only if $\alpha > \ln 2/\ln 10 = 0.3010\ldots$

3. The series $\sum n^{-\alpha}$ over all natural numbers without odd digits in their decimal expansion converges only for $\alpha > \ln 5/\ln 10 = 0.69807\ldots$

4. If the frequency of exactly one digit is fixed, we have $\alpha_0 = \log_d(d-1) = \ln(d-1)/\ln d$, which for $d = 10$ is 0.9542...

5. If all n_i's are zero, then inequality (1) simplifies and becomes

$$S(\alpha, \mathbf{a}, (0, \ldots, 0)) \leq \Big(1 + \frac{d^\alpha}{k + d^\alpha - d}\Big)\Big(\sum_{n=1}^{d-1} n^{-\alpha} - \sum_{p=1}^{k} a_p^{-\alpha}\Big). \tag{2}$$

6. If we set $d = 10, k = 1, a_1 = 9, n_1 = 0$ in (2) we are back to Kempner's original series (all reciprocals without 9's), but for general α. In this case we get from (2) for $\alpha = 1$

$$S(1, \mathbf{a},0) \leq 11 \times \Big(1 + \frac{1}{2} + \cdots + \frac{1}{8}\Big) = 29.8964\ldots,$$

and for $\alpha = 0.96$ (closer to the threshold $\alpha_0 = 0.9542\ldots$) the upper bound increases to

$$\Big(1 + \frac{10^{0.96}}{10^{0.96} - 9}\Big)\sum_{n=1}^{8} n^{-0.96} = 139.18\ldots$$

7. Finally, let $\alpha = 1, k = 1, a_1 \in \{1, \ldots, d-1\}$. Then we get the series of the reciprocals having the digit a_1 exactly n_1 times in their d-ary expansion. Inequality (1) becomes

$$S(1, a_1, 0) \leq (1 + d)\Big(\sum_{n=1}^{d-1} n^{-1} - a_1^{-1}\Big)$$

$$S(1, a_1, 1) \leq a_1^{-1} + d\Big(\sum_{n=1}^{d-1} n^{-1} - a_1^{-1}\Big) + a_1^{-1}d = a_1^{-1} + d\sum_{n=1}^{d-1} n^{-1}$$

$$S(1, a_1, n_1) \leq d\sum_{n=1}^{d-1} n^{-1}, \quad \text{if } n_1 > 1.$$

2 Proof of divergence

Fix $\alpha \leq \log_d(d-k)/\log d$, $\mathbf{a}$ and $\mathbf{n}$. Let $A_j = \{n \in A(\mathbf{a}, \mathbf{n}) \mid 0 \leq n < d^j\}$, $j \geq 1$. Then

$$\begin{aligned} S(\alpha, \mathbf{a}, \mathbf{n}) &= \lim_{j\to\infty} \sum_{1\leq n<d^{j+1}, n\in A(\mathbf{a},\mathbf{n})} n^{-\alpha} \\ &= \sum_{i=1}^{\infty} \sum_{d^i\leq n<d^{i+1}, n\in A(\mathbf{a},\mathbf{n})} n^{-\alpha} \\ &\geq \sum_{i=1}^{\infty} d^{-\alpha(i+1)}[\mathrm{card}A_{i+1} - \mathrm{card}A_i]. \end{aligned} \tag{3}$$

We have to determine $\text{card}A_j$. Any $n \in \{0, \ldots, d^j - 1\}$ can be represented by its d-ary expansion

$$n = \sum_{i=0}^{j-1} m_i d^i$$

with digits $m_i \in \{0, \ldots, d-1\}$. If $N = \sum_{p=1}^{k} n_p > j$, there are not enough digits for an $n < d^j$ to be in $A(\mathbf{a}, \mathbf{n})$. Thus,

$$\text{card}A_j = 0 \text{ if } N > j.$$

Now let $N \leq j$. The d-ary expansion of any number in A_j can be created in two steps:

1. First, choose N indices, say $i_1, \ldots, i_N$, from $\{0, \ldots, j-1\}$ and then digits $m_{i_1}, \ldots, m_{i_N}$ of which exactly n_i are equal to a_i for $i = 1, \ldots, k$. This can be done in

$$\binom{j}{N}\binom{N}{n_1}\binom{N-n_1}{n_2}\cdots\binom{N-n_1-\cdots-n_{k-1}}{n_k} = \frac{j!}{(j-N)!n_1!\cdots n_k!}$$

ways.

2. Second, choose the remaining $j - N$ digits not equal to any of the a_i. For any choice in the first step this can be done in $(d-k)^{j-N}$ ways.

Combining the two steps it follows that

$$\text{card}A_j = \frac{j!(d-k)^{j-N}}{(j-N)!n_1!\cdots n_k!}, \quad \text{if } N \leq j. \tag{4}$$

Inserting (4) in (3) we can continue the calculation in (3) as follows:

$$\begin{aligned} S(\alpha, \mathbf{a}, \mathbf{n}) &\geq \sum_{i=N}^{\infty} d^{-\alpha(i+1)} \left[\frac{(i+1)!(d-k)^{i+1-N}}{(i+1-N)!n_1!\cdots n_k!} - \frac{i!(d-k)^{i-N}}{(i-N)!n_1!\cdots n_k!} \right] \\ &= \lim_{j\to\infty} \frac{N!}{d^\alpha (d-k)^N n_1!\cdots n_k!} \sum_{i=N}^{j} \left[d^\alpha \binom{i+1}{N}\left(\frac{d-k}{d^\alpha}\right)^{i+1} - \binom{i}{N}\left(\frac{d-k}{d^\alpha}\right)^{i} \right] \\ &\geq \lim_{j\to\infty} \frac{N!}{d^\alpha (d-k)^N n_1!\cdots n_k!} \sum_{i=N}^{j} \left[\binom{i+1}{N}\left(\frac{d-k}{d^\alpha}\right)^{i+1} - \binom{i}{N}\left(\frac{d-k}{d^\alpha}\right)^{i} \right] \\ &= \lim_{j\to\infty} \frac{N!}{d^\alpha (d-k)^N n_1!\cdots n_k!} \left[\binom{j+1}{N}\left(\frac{d-k}{d^\alpha}\right)^{j+1} - \left(\frac{d-k}{d^\alpha}\right)^{N} \right] \\ &= \infty. \end{aligned} \tag{5}$$

In the third step of (5) we omitted the factor d^α; thereafter we used a telescoping sum. For the last equality note that $(d-k)/d^\alpha \geq 1$ follows from the assumption $\alpha \leq \log_d(d-k)$.

3 Derivation of the upper bound

Now let $\alpha > \log_d(d-k)$. To prove inequality (1), we define

$$\mathbf{n}^{(p)} = (n_1, \dots, n_{p-1}, n_p - 1, n_{p+1}, \dots, n_k), \quad p = 1, \dots, k$$
$$D = \{1, \dots, d-1\} \backslash \{d_1, \dots, d_k\}.$$

We split the series into subseries according to the first significant digit of n:

$$\begin{aligned}
S(\alpha, \mathbf{a}, \mathbf{n}) &= s(\alpha, \mathbf{a}, \mathbf{n}) + \sum_{i=1}^{\infty} \sum_{\substack{d^i \le n < d^{i+1}, \\ n \in A(\mathbf{a}, \mathbf{n})}} n^{-\alpha} \\
&= s(\alpha, \mathbf{a}, \mathbf{n}) + \sum_{i=1}^{\infty} \left[\sum_{\substack{l \in D, 0 \le m < d^i, \\ m \in A(\mathbf{a}, \mathbf{n})}} (l d^i + m)^{-\alpha} + \sum_{p=1}^{k} \sum_{\substack{0 \le m < d^i, \\ m \in A(\mathbf{a}, \mathbf{n}^{(p)})}} (a_p d^i + m)^{-\alpha} \right] \\
&\le s(\alpha, \mathbf{a}, \mathbf{n}) + \sum_{l \in D} l^{-\alpha} \sum_{i=1}^{\infty} d^{-i\alpha} \mathrm{card} A_i + \sum_{p=1}^{k} a_p^{-\alpha} \sum_{i=1}^{\infty} d^{-i\alpha} \mathrm{card} A_i^{(p)}, \qquad (6)
\end{aligned}$$

where

$$A_i^{(p)} = \{n \in A(\mathbf{a}, \mathbf{n}^{(p)}) \mid 0 \le n < d^i\}.$$

Note that $\mathrm{card} A_i^{(p)} = 0$ if $n_p = 0$, because in this case $A(\mathbf{a}, \mathrm{n}^{(p)})$ is empty.

Using (4) we obtain

$$\begin{aligned}
\sum_{i=1}^{\infty} d^{-i\alpha} \mathrm{card} A_i &= \sum_{i=N}^{\infty} d^{-i\alpha} \frac{i!(d-k)^{i-N}}{(i-N)! n_1! \cdots n_k!} \\
&= \frac{N!(d-k)^{-N}}{n_1! \cdots n_k!} \sum_{i=N}^{\infty} \binom{i}{N} \left(\frac{d-k}{d^\alpha} \right)^i \\
&= \frac{N!(d-k)^{-N}}{n_1! \cdots n_k!} \left(\frac{d-k}{d^\alpha} \right)^N \left(1 - \frac{d-k}{d^\alpha} \right)^{-N-1} \\
&= \frac{N! d^\alpha}{n_1! \cdots n_k! (k + d^\alpha - d)^{N+1}}. \qquad (7)
\end{aligned}$$

Similarly,

$$\sum_{i=1}^{\infty} d^{-i\alpha} \mathrm{card} A_i^{(p)} = \frac{(N-1)! d^\alpha n_p}{n_1! \cdots n_k! (k + d^\alpha - d)}. \qquad (8)$$

Inserting (7) and (8) in (6) we obtain the upper bound (1).

References

[1] R. Baillie, Sums of reciprocals of integers missing a given digit. *Amer. Math. Monthly* **86** (1979), 372–374.

[2] R. Baillie, Summing the curious series of Kempner and Irwin. (2013), *arXiv* 0806.4410v3.

[3] H.-J. Fischer, Die Summe der Reziproken der natürlichen Zahlen ohne Ziffer 9. *Elem. Math.* **48** (1993), 100–106.

[4] F. Irwin, A curious converging series. *Amer. Math. Monthly* **23** (1914), 149–152.

[5] A. Kempner, A curious converging series. *Amer. Math. Monthly* **21** (1914), 48–50.

[6] T. Schmelzer and R. Baillie, Summing a curious, slowly converging series. *Amer. Math. Monthly* **115** (2008), 525–540.

[7] W. Stadje, Konvergenz von Teilen der harmonischen Reihe. *Elem. Math.* **46** (1991), 51–54.

[8] R.-O. Vilceanu, On the series of Kempner–Irwin type. *Ann. Univ. Craiova, Math. Comp. Sci. Ser.* **36** (2009), 112–121.

Wolfgang Stadje
Institut für Mathematik
Universität Osnabrück
D-49069 Osnabrück, Germany
e-mail: wstadje@uos.de