Elem. Math. 70 (2015) 144 – 152
0013-6018/15/040144-9
DOI 10.4171/EM/289

Elemente der Mathematik

The mystery of the number 1089 – how Fibonacci numbers come into play

Ehrhard Behrends

Ehrhard Behrends ist Professor im Ruhestand. Bis 2014 hat er an der Freien Universität Berlin gearbeitet. Seine Spezialgebiete sind Funktionalanalysis und Wahrscheinlichkeitstheorie. Seit Jahren ist er auch in der Popularisierung der Mathematik aktiv. Mehrfach hat er Arbeiten zum mathematischen Hintergrund von Zaubertricks publiziert. Seit 2014 ist er Mitglied des Magischen Zirkels von Deutschland.

Choose any positive number a with three digits where the last digit is smaller than the first one. Reverse the order of the digits and calculate a minus the reverse of a. Call the result b and add to b the reverse of b. The result will always be 1089.

As an example consider $a = 745$. First we calculate $745 - 547$ to obtain $b = 198$. And really one has $198 + 891 = 1089$ as predicted[1]. One can prove this fact by using very elementary arithmetic.

It has often been transformed to a mathematical prediction trick. One finds it in many books concerned with magical tricks with a mathematical background, and GOOGLE offers more than 1.6 million links when asking for "1089 trick".

[1]Note that b has to be considered as a three digit number when we reverse it: for example, the reverse of 011 is 110. If one wants to avoid this somehow artificial extra rule one could restrict oneself to numbers a where the first digit minus the last digit is larger than one.

Wenn man eine dreistellige Zahl xyz (mit $x > z$) spiegelt und das Ergebnis zyx von xyz abzieht, erhält man eine Zahl def. Überraschenderweise ist dann immer $def + fed = 1089$. Dieses Phänomen wird oft für einen Zaubertrick verwendet. In der vorliegenden Arbeit wird untersucht, was passiert, wenn man statt mit einer dreistelligen Zahl mit einer n-stelligen Zahl beginnt, wobei n ganz beliebig sein kann. Es ist dann nicht mehr richtig, dass man immer das gleiche Endergebnis erhält. In der Regel werden – je nach Startzahl – am Ende verschiedene Zahlen herauskommen, die Anzahl möglicher Endergebnisse ist aber immer bemerkenswert klein. Zwei Tatsachen sind überraschend. Erstens treten bei der Formulierung des Ergebnisses die Fibonacci-Zahlen auf. Und zweitens ist der technische Aufwand, den man für den Beweis aufbieten muss, sehr viel höher, als man es bei so einem Problem aus der elementaren Arithmetik vermuten würde.

The aim of the present note is to investigate what happens if one replaces three digit numbers by numbers of arbitrary length. More precisely we fix an $n \geq 2$, and we will consider n-digit numbers $a = a_1a_2 \ldots a_n$ with $a_i \in \{0, \ldots, 9\}$ and $a_1 > a_n$. Then we calculate $a_1 \ldots a_n - a_n \ldots a_1$, and we write this positive number as $b_1 \ldots b_n$. Finally we calculate $b_1 \ldots b_n + b_n \ldots b_1$, this number will be called $\phi_n(a)$.

Suppose, e.g., that we consider in the case $n = 6$ the number $a = 242141$. Then $b_1 \ldots b_6 = 242141 - 141242 = 100899$, and $\phi_6(a) = 100899 + 998001 = 1098900$. In the case $n = 4$ and $a = 8007$ we calculate as follows: $8007 - 7008 = 0999$, and $0999 + 9990 = 10989$; note that always (as in the case $n = 3$) $b_1 \ldots b_n$ has to be considered as an n-digit number, leading zeros have to be taken into account when passing from $b_1 \ldots b_n$ to $b_n \cdots b_1$.

Our investigations started with the observation that all $\phi_3(a_1a_2a_3)$ equal 1089 when $a_1a_2a_3$ runs through the positive numbers with 3 digits such that $a_1 > a_3$. It is *not* true, however, that also for larger n all $\phi_n(a_1 \cdots a_n)$ coincide. But we will be able to show that there are always surprisingly few different numbers in the range of ϕ_n and that – completely unexpectedly – Fibonacci numbers enter the scene.

We will also treat *another generalization*: up to now we worked in the decimal system, but one could ask the same question if the numbers under consideration are represented otherwise. What happens, e.g., with dyadic numbers or with numbers represented in the hexadecimal system when we apply the same rules? In the sequel the number $B \in \{2, 3, \ldots\}$ will be fixed, and we will expand integers in the B-adic system. Those readers who are not interested in the general approach should replace B by the number 10 in the sequel to stay in the well-known decimal system.

Here are the relevant definitions:

- $I_{B,n} = \{0, \ldots, B - 1\}^n$ denotes the set of B-adic expansions of the integers m with $0 \leq m \leq B^n - 1$. The elements of $I_{B,n}$ will be written as $(a_1 \ldots a_n)_B$. For example, $(20045)_{10}$ is "really" the number 20045, whereas $(10011)_2$ is the dyadic expansion of the number 19.
- $I^*_{B,n}$ stands for the $(a_1 \ldots a_n)_B \in I_{B,n}$ such that $a_1 > a_n$.
- The map $\rho_{B,n} : I_{B,n} \to I_{B,n}$ reverses the order: $\rho_{B,n} : (a_1 \ldots a_n)_B \mapsto (a_n \ldots a_1)_B$.
- $\delta_{B,n} : I^*_{B,n} \to I_{B,n}$ maps an $(a_1 \ldots a_n)_B$ to the B-adic expansion of the difference $(a_1 \ldots a_n)_B$ minus $\rho_{B,n}\big((a_1 \ldots a_n)_B\big)$.
- $\sigma_{B,n} : I_{B,n} \to I_{B,n+1}$ maps a $(b_1 \ldots b_n)_B$ to the B-adic expansion of the sum of $(b_1 \ldots b_n)_B$ and $\rho_{B,n}\big((b_1 \ldots b_n)_B\big)$; it can happen that this number has $n + 1$ B-adic digits. Example: $\sigma_{5,3}\big((243)_5\big) = (243)_5 + (342)_5 = (1140)_5$.
- And finally, $\phi_{B,n} : I^*_{B,n} \to I_{B,n+1}$ is defined by $\phi_{B,n} := \sigma_{B,n} \circ \delta_{B,n}$. (Note that $\phi_{10,n}$ coincides with the map ϕ_n that was introduced above.)

Admittedly these are rather technical definitions, but they are necessary for a formal generalization of the rule that we have described above when introducing the 1089 trick.

How many elements are there in the range of $\phi_{B,n}$? Here is our main result:

Theorem. *Depending on whether the integer $n \geq 2$ is even or odd we write n as $2r$ or $2r + 1$. The sequence $F_1, F_2, F_3, \ldots$ denotes the usual Fibonacci sequence* $1, 1, 2, 3, 5, \ldots$

Then precisely F_{2r} different numbers will occur as $\phi_{B,n}\big((a_1 \cdots a_n)_B\big)$ when $(a_1 \cdots a_n)_B$ runs through the elements of $I^*_{B,n}$: there is $1 = F_2$ number for $n = 2$ and $n = 3$ (this corresponds to the original trick), the cases $n = 4$ and $n = 5$ give rise to $3 = F_4$ different numbers etc.

As two immediate consequences we note:

- The number of possible candidates for the $\phi_{B,n}\big((a_1 \cdots a_n)_B\big)$ does not depend on B.
- This number is tiny when compared with the elements of $I^*_{B,n}$. The proportion for even n is of order $F_n/B^n \approx (\varphi/B)^n$ where $\varphi = (1 + \sqrt{5})/2 = 1.618\ldots$ is the golden ratio.

The rest of this note is devoted to the proof of this theorem. It will depend on an elementary but nevertheless surprisingly involved analysis of the arithmetic that is used to transform $(a_1 \cdots a_n)_B \in I^*_{B,n}$ to $\phi_{B,n}\big((a_1 \cdots a_n)_B\big)$. At the end of this note one finds proposals how to use our result for a *mathematical magical trick*.

Reminder 1: differences. Most readers will be surprised to be reminded of some very elementary school arithmetic in a scientific mathematical paper, but this will be necessary to explain a definition that will be important for our investigations.

Carries will play a crucial role here, three variants will be used in the sequel (the t_k, the u_k, and the v_k).

Let $e = (e_1 \cdots e_n)_B$ and $d = (d_1 \cdots d_n)_B$ in $I_{B,n}$ with $e > d$ be given. How does one calculate $e - d$ in B-adic expansion? One works backwards from the last digit to the first one, sometimes – when calculating the kth digit – it might be necessary to "borrow" a 1 from the $(k-1)$th digit. (It should be noted that school children are taught different strategies: in Germany, e.g., one adds a "1" to d_{k-1} whereas in the USA one "borrows" a 1 from e_{k-1}.)

The first family of carries $t_{n+1}, t_n, t_{n-1}, \ldots, t_1$ is defined as follows: $t_{n+1} := 0$ and $t_k := 0$ if $e_k \geq d_k + t_{k+1}$ and $t_k := 1$ otherwise. Then the kth digit of $e - d$ in B-adic expansion is $Bt_k + e_k - (d_k + t_{k+1}) \in \{0, 1, \ldots, B-1\}$ $(k = n, n-1, \ldots, 1)$. We will use the notation $C(e, d) := t_1 \cdots t_n$.

Here are two examples for the usual decimal system to illustrate this definition: $(5553)_{10} - (1223)_{10}$ leads to $t_1t_2t_3t_4 = 0000$, i.e., $C\big((5553)_{10}, (1223)_{10}\big) = 0000$. And $(555370)_{10} - (499999)_{10}$ yields $C\big((555370)_{10}, (499999)_{10}\big) = 011111$.

Of particular interest will be the $t_1 \ldots t_n$ when we calculate the difference $\delta_{B,n}(a) = a - \rho_{B,n}(a)$ for $a = (a_1 \cdots a_n)_B \in I^*_{B,n}$.

By $\tau_{B,n} : I^*_{B,n} \to \{0, 1\}^n$ we denote the map that associates to $a = (a_1 \cdots a_n)_B \in I^*_{B,n}$ the pattern $C\big((a_1 \cdots a_n)_B, (a_n \cdots a_1)_B\big)$. (So that, e.g., $\tau_{10,7}\big((4555552)_{10}\big) = 0111111$.) $T_{B,n} \subset \{0, 1\}^n$ stands for the range of $\tau_{B,n}$.

Our *strategy to prove the theorem* will be as follows: first we will determine in Lemma 2 the cardinality of $T_{B,n}$, and then we will show in Lemma 3 that there is a bijection between $T_{B,n}$ and the range of $\phi_{B,n}$.

The following facts can easily be verified:

Lemma 1. *Fix* $a = (a_1 \cdots a_n)_B \in I^*_{B,n}$ *and put* $t_1 \dots t_n := \tau_{B,n}(a)$.

(i) $t_1 = 0$ *and* $t_n = 1$.

(ii) *If* $a_k > a_{n-k+1}$ *then* $t_k = 0$*; if* $a_k < a_{n-k+1}$ *then* $t_k = 1$*; if* $a_k = a_{n-k+1}$ *then* $t_k = t_{k+1}$ $(k = 1, \dots, n)$.

(iii) *For* $k = 1, \dots, n$ *the kth digit of* $\delta_{B,n}(a)$ *is* $t_k B + a_k - (a_{n-k+1} + t_{k+1})$ *; here, as above, we put* $t_{n+1} := 0$.

In order to be able to calculate the cardinality of $T_{B,n}$ by a recursion formula we will need some further definitions:

1. $T^0_{B,n}$ (resp. $T^1_{B,n}$) denotes the collection of the $t_1 \cdots t_n \in T_{B,n}$ such that $t_2 = 0$ (resp. $t_2 = 1$). And Ψ_n (resp. Ψ^0_n resp. Ψ^1_n) stands for the cardinality of $T_{B,n}$ (resp. $T^0_{B,n}$ resp. $T^1_{B,n}$). We note that, by part (ii) of the preceding lemma, $T_{B,n}$ does not depend on B.
2. A map $\mu_{B,n} : I^*_{B,n} \to \{0, 1\}^n$ (a variant of $\tau_{B,n}$) is defined by $a = (a_1 \cdots a_n)_B \mapsto u_1 \cdots u_n := C\big((a_1 \cdots a_n)_B, (a_n \cdots a_2 0)_B\big)$: before calculating the difference of a and the reverse of a the last digit of this reverse is changed to zero.

It is clear that always $u_n = 0$ and $u_1 = 0$ hold.

3. $M_{B,n}$ denotes the range of $\mu_{B,n}$, and $M^0_{B,n}$ (resp $M^1_{B,n}$) is the collection of the $u_1 \cdots u_n \in M_{B,n}$ such that $u_2 = 0$ (resp. $u_2 = 1$). And Φ_n (resp. Φ^0_n resp. Φ^1_n) denotes the cardinality of $M_{B,n}$ (resp. $M^0_{B,n}$ resp. $M^1_{B,n}$).

Here are some concrete calculations. First we will restrict ourselves to the case of even n, we will write $n = 2r$.

1) We start with $n = 4$. For the calculation of $\tau_{B,4}(a)$ for a certain $a = (a_1 a_2 a_3 a_4) \in I^*_{B,4}$ one only needs to know whether $a_2 < a_3$, $a_2 = a_3$ or $a_2 > a_3$. And therefore, if one wants to identify the elements of $T_{B,4}$, one only has to treat three examples. We choose $(\beta 0 \beta 0)_B$, $(\beta 000)_B$ and $(\beta\beta 00)_B$, where $\beta := B - 1$. The following table shows these a together with the associated $\tau_{B,4}(a)$:

a	$(\beta 0 \beta 0)_B$	$(\beta 000)_B$	$(\beta\beta 00)_B$
$\tau_{B,4}$	0101	0111	0011

It follows that $\Psi^0_4 = 1$, $\Psi^1_4 = 2$ and $\Psi_4 = 3$.

And here is the corresponding table for $M_{B,4}$:

a	$(\beta 0 \beta 0)_B$	$(\beta 000)_B$	$(\beta\beta 00)_B$
$\mu_{B,4}$	0100	0000	0010

We conclude that $\Phi^0_4 = 2$, $\Phi^1_4 = 1$ and $\Phi_4 = 3$.

2) Next we consider the case $n = 6$. This time 9 different $a \in I^*_{B,6}$ have to be treated in order to exhaust all possibilities: $a_2 <, =, > a_5$ and $a_3 <, =, > a_4$. In the table one sees our choice of a and the corresponding $\tau_{B,6}$:

a	$(\beta 00\beta\beta 0)_B$	$(\beta 000\beta 0)_B$	$(\beta 0\beta 0\beta 0)B$	$(\beta 00\beta 00)_B$	$(\beta 00000)_B$
$\tau_{B,6}$	011001	010001	010101	011011	011111

a	$(\beta 0\beta 000)_B$	$(\beta\beta 0\beta 00)_B$	$(\beta\beta 0000)_B$	$(\beta\beta\beta 000)_B$
$\tau_{B,6}$	000111	001011	001111	000111

Thus $\Psi^0_6 = 3$, $\Psi^1_6 = 5$ and $\Psi_6 = 8$; note that the pattern 000111 appears twice in this table, it has to be counted only once.

The range of $\mu_{B,6}$ contains the following elements:

a	$(\beta 00\beta\beta 0)_B$	$(\beta 000\beta 0)_B$	$(\beta 0\beta 0\beta 0)B$	$(\beta 00\beta 00)_B$	$(\beta 00000)_B$
$\mu_{B,6}$	011000	010000	010100	011000	000000

a	$(\beta 0\beta 000)_B$	$(\beta\beta 0\beta 00)_B$	$(\beta\beta 0000)_B$	$(\beta\beta\beta 000)_B$
$\mu_{B,6}$	000100	001010	001110	000110

It follows that $\Phi^0_6 = 5$, $\Phi^1_6 = 3$ and $\Phi_6 = 8$.

Lemma 2.

(i) *The following recursion formulas hold for $r \geq 1$:*

$$\Psi^0_{2(r+1)} = \Psi_{2r},\ \Psi^1_{2(r+1)} = \Psi^1_{2r} + \Phi_{2r},\ \Phi^0_{2(r+1)} = \Phi^0_{2r} + \Psi_{2r},\ \Phi^1_{2(r+1)} = \Phi_{2r}.$$

(ii) $\Psi_{2r} = F_{2r}$, *where F_{2r} denotes the $2r$th element of the Fibonacci sequence F_1, F_2, $\cdots = 1, 1, 2, 3, 5, 8, \ldots$*

(iii) *Write $n = 2r$ if n is even and $n = 2r + 1$ if n is odd. Then $T_{B,n}$ has F_{2r} elements.*

Proof. (i) It will be convenient to write an $\tilde{a} \in I^*_{B,2(r+1)}$ in the form

$$\tilde{a} = (a_1 \alpha a_2 \cdots a_{2r-1} \alpha' a_{2r})_B$$

with $\alpha, \alpha' \in \{0, \ldots, B-1\}$ (so that, e.g., a_2 denotes the *third* digit in $\tilde{a}$). Put $a := (a_1 \cdots a_{2r})_B \in I^*_{B,2r}$, $t_1 \cdots t_{2r} := \tau_{B,2r}(a)$ and $u_1 \cdots u_{2r} := \mu_{B,2r}(a)$. Then it follows from elementary arithmetic that:

- If $\alpha < \alpha'$ then $\tau_{B,2(r+1)}(\tilde{a}) = 01u_2 \cdots u_{2r-1}01$ and $\mu_{B,2(r+1)}(\tilde{a}) = 01u_2u_3 \cdots u_{2r-1}00$;
- If $\alpha = \alpha'$ then $\tau_{B,2(r+1)}(\tilde{a}) = 0t_2t_2t_3 \cdots t_{2r-1}11$ and $\mu_{B,2(r+1)}(\tilde{a}) = 0u_2u_2u_3 \cdots u_{2r-1}00$;
- If $\alpha > \alpha'$ then $\tau_{B,2(r+1)}(\tilde{a}) = 00t_2 \cdots t_{2r-1}11$ and $\mu_{B,2(r+1)}(\tilde{a}) = 00t_2t_3 \cdots t_{2r-1}10$;

The recursion formulas can now be deduced easily:

a) How many elements are there in $T^0_{B,2(r+1)}$? They are generated only when $\alpha > \alpha'$ or when $\alpha = \alpha'$. In the second case only the $\tilde{a}$ with $t_2 = 0$ contribute, but these are already part of the collection generated by $\alpha > \alpha'$. This proves $\Psi^0_{2(r+1)} = \Psi_{2r}$.

b) Only the $\tilde{a}$ with $\alpha < \alpha'$ and the $\tilde{a}$ with $\alpha = \alpha'$ and $t_2 = 1$ count for $\Psi^1_{2(r+1)}$, and all these patterns are different: the last but one digit in the first family (from $\alpha < \alpha'$) is 0 whereas it is 1 in the second. This shows that $\Psi^1_{2(r+1)} = \Psi^1_{2r} + \Phi_{2r}$.

c) and d) The recursion formulas for $\Phi^0_{2(r+1)}$ and $\Phi^1_{2(r+1)}$ are justified in a similar way.

(ii) By the above calculations we know that $\Psi^0_4 = \Phi^1_4 = F_2$, $\Psi^1_4 = \Phi^0_4 = F_3$ and $\Psi_4 = \Phi_4 = F_4$. It follows easily from the relation $F_k + F_{k+1} = F_{k+2}$ and the recursion formulas from (i) that always $\Psi^0_{2r} = \Phi^1_{2r} = F_{2r-2}$, $\Psi^1_{2r} = \Phi^0_{2r} = F_{2r-1}$ and $\Psi_{2r} = \Phi_{2r} = F_{2r}$. This proves the claim for $r \geq 2$; for $r = 1$ it is trivially true.

(iii) The case of even $n = 2r$ is covered by (ii) since Ψ_{2r} counts the elements of $T_{B,2r}$. Now let $n = 2r + 1$ be odd. By Lemma 1 (ii) we know that any $t_1 \cdots t_n \in T_{B,n}$ satisfies $t_{r+1} = t_{r+2}$ since $a_k = a_{n-k+1}$ for $k = r + 1$. Therefore $t_1 \cdots t_r t_{r+1} t_{r+2} \cdots t_n \mapsto t_1 \cdots t_r t_{r+2} \cdots t_n$ is a bijection between $T_{B,2r+1}$ and $T_{B,2r}$. □

Reminder 2: sums. Summation in B-adic expansion is easier than subtraction. Let $d = (d_1 \dots d_n)_B$ and $e = (e_1 \dots e_n)_B$ in $I_{B,n}$ be given. Denote by υ_k the carry that occurs when calculating the kth digit of $d + e$. This means that we define $\upsilon_1, \dots, \upsilon_n, \upsilon_{n+1}$ recursively by $\upsilon_{n+1} := 0$, and $\upsilon_k = 1$ (resp. $\upsilon_k := 0$) if $d_k + e_k + \upsilon_{k+1} \geq B$ (resp. $d_k + e_k + \upsilon_{k+1} < B$); $k = n, n-1, \dots, 1$. Then the B-adic expansion of $d + e$ is given by $\upsilon_1 c_1 \dots c_n$, where $c_k := \upsilon_{k+1} + d_k + e_k - \upsilon_k B$ for $k = 1, \dots, n$.

It will be convenient for us to have *an intermediate step* in our calculation: first we calculate the numbers $R_k := d_k + e_k \in \{0, \dots, 2B-2\}$ $(k = 1, \dots, n)$, and from these we determine the B-adic expansion of $d+e$. For example, $(34201)_5 + (44033)_5$ is calculated as

$$(34204)_5 + (44033)_5 \mapsto (7, 8, 2, 3, 7) \mapsto (133242)_5;$$

here the carries are $\upsilon_1 \upsilon_2 \upsilon_3 \upsilon_4 \upsilon_5 = 11001$.

Of particular interest will be the case $d = (b_1 \cdots b_n)_B = \delta_{B,n}(a)$ and $e = (b_n \cdots b_1)_B$ for $a = (a_1 \cdots a_n)_B \in I^*_{B,n}$. Let such an a be given. We already know (Lemma 1 (iii)) that the kth digit of $b = (b_1 \cdots b_n)_B := \delta_{B,n}(a)$ is $t_k B + a_k - (a_{n-k+1} + t_{k+1})$. Therefore b_k plus the kth digit of $\rho_{B,n}(b)$ is

$$\begin{aligned} R_k &:= b_k + b_{n-k+1} \\ &= t_k B + a_k - (a_{n-k+1} + t_{k+1}) + t_{n-k+1} B + a_{n-k+1} - (a_k + t_{n-k+2}) \\ &= (t_k + t_{n-k+1}) B - (t_{k+1} + t_{n-k+2}). \end{aligned}$$

(This is a crucial observation: the $R_1, \dots, R_n$ only depend on the t_k and not on the a_k.) In order to calculate $\phi_{B,n}(a)$ as a B-adic number it remains to work from the right to the left:

we define the υ_k as the carries when determining $(b_1 \cdots b_n)_B + (b_n \cdots b_1)_B$ as above[2]. Then with $c_k := R_k + \upsilon_{k+1} - \upsilon_k B$ one has $\phi_{B,n}(a) = (\upsilon_1 c_1 \ldots c_n)_B$.

Here is an example, we consider $a = (5677321)_{10}$. Then $\delta_{10,7}(a) = (4439556)_{10}$ and $(R_1, \ldots, R_7) = (10, 9, 8, 18, 8, 9, 10)$. Consequently $\phi_{10,7}(a) = (10998900)_{10}$ with $\upsilon_1 \cdots \upsilon_7 = 1001011$.

Lemma 3.

(i) $R_k = R_{n-k+1}$, *and* $R_k \in \{0, B-2, B-1, B, 2B-2\}$ *for all* k.

(ii) *The map* $t_1 \cdots t_n \mapsto (R_1, \ldots, R_n)$ (*from* $T_{B,n}$ *to* $\{0, B-2, B-1, B, 2B-2\}^n$) *is one to one.*

(iii) *The map* $(R_1, \ldots, R_n) \mapsto \upsilon_1 c_1 \cdots c_n$ (*from the* $(R_1, \ldots, R_n)$ *that are generated by the* $(a_1 \cdots a_n)_B \in I^*_{B,n}$ *to* $I_{B,n+1}$) *is one to one.*

Proof. (i) The symmetry is a consequence of the definition: $R_k = b_k + b_{n-k+1}$ for $k = 1, \ldots, n$. That R_k lies in $\{0, B-2, B-1, B, 2B-2\}$ follows from the formula $R_k = (t_k + t_{n-k+1})B - (t_{k+1} + t_{n-k+2})$ and the fact that R_k is the sum of two elements in $\{0, 1, \ldots, B-1\}$.

(ii) We have to show that it is possible to reconstruct $t_1 \cdots t_n$ from $(R_1, \ldots, R_n)$. Always $t_1 = 0 = t_{n+1}$ and $t_n = 1$ hold so that

$$R_1 = (t_1 + t_n)B - (t_2 + t_{n+1}) = B - t_2.$$

In this way we have identified t_1, t_2, t_n. The remaining t_k will be found by working recursively "inwards"; from t_1, t_2, t_n to $t_1, t_2, t_3, t_{n-1}, t_n$, then to $t_1, t_2, t_3, t_4, t_{n-2}, t_{n-1}, t_n$ etc.

Suppose that we know for some $k \geq 2$ the $t_1, \ldots, t_k, t_{n-k+2}, \ldots, t_n$. What can be said about t_{k+1} and t_{n-k+1}? We consider four cases separately.

Case 1*:* $t_k = t_{n-k+2} = 0$. In this case $R_k = (t_k + t_{n-k+1})B - (t_{k+1} + t_{n-k+2}) = t_{n-k+1}B - t_{k+1}$ holds, where R_k is known. t_{k+1} and t_{n-k+1} can now be identified with the help of (i): the number R_k is one of the numbers $B, B-1, 0$, and this yields $t_{n-k+1} = 1, t_{k+1} = 0$ or $t_{n-k+1} = 1, t_{k+1} = 1$ or $t_{n-k+1} = t_{k+1} = 0$, respectively.

Case 2*:* $t_k = 1, t_{n-k+2} = 0$. Then $R_k = (1 + t_{n-k+1})B - t_{k+1}$. R_k equals $B, B-1, 2B$ or $2B-1$, and in each case one can reconstruct t_{k+1} and t_{n-k+1}. (For example, if $R_k = 2B$, then necessarily $t_{n-k+1} = 1$ and $t_{k+1} = 0$.)

Case 3 ($t_k = 0, t_{n-k+2} = 1$) and case 4 ($t_k = t_{n-k+2} = 1$) are treated in a similar way. This proves (ii).

(iii) How can one find $R_1, \ldots, R_k$ if $\upsilon_1 c_1 \cdots c_n$ are known? We have $R_n = B - t_2$ so that $R_n = B$ or $R_n = B - 1$. Thus it follows in the case $c_n = 0$ that $R_n = B$ and $\upsilon_n = 1$ whereas $c_n = B - 1$ yields $R_n = B - 1$ and $\upsilon_n = 0$. The number υ_1 is also known by assumption so that we can start our recursion with the known numbers $R_1 = R_n$ and

[2] I.e., $\upsilon_{n+1} := 0$, and $\upsilon_k = 1$ (resp. $\upsilon_k = 0$) if $R_k + \upsilon_{k+1} \geq B$ (resp. $< B$).

υ_1, υ_n. As in the proof of (ii) we work from the extreme left and right indices to the inner ones: from $1, n$ to $1, 2, n-1, n$ etc.

Suppose that $R_1(= R_n), R_2(= R_{n-1}), \ldots, R_k(= R_{n-k+1})$ and $\upsilon_1, \ldots, \upsilon_k, \upsilon_{n-k+1}, \ldots, \upsilon_n$ are already found. We will determine $R_{k+1}(= R_{n-k})$ as well as υ_{k+1} and υ_{n-k}. Write the B-adic expansion of R_{k+1} as $(\alpha\alpha')_B$; here $\alpha \in \{0, 1\}$ and $\alpha' \in \{0, B-1, B-2\}$.

Step 1*: First we identify* υ_{k+1}. As an example consider a case where $R_k = c_k = B - 1$. Then υ_{k+1} necessarily is 0 since $\upsilon_{k+1} = 1$ would imply $c_k = 0$. Similarly $\upsilon_{k+1} = 0$ must hold whenever $R_k = c_k$ or – in situations where $R_k = (10)_B$ or $R_k = (1, B-2)_B$ – when c_k equals the second (counted from left to right) B-adic digit of R_k. In all other cases one knows that $\upsilon_{k+1} = 1$.

Step 2*: We determine* α'. By assumption we know υ_{n-k+1}. If this number is zero then $\alpha' = c_{n-k}$. In the case $\upsilon_{n-k+1} = 1$ we consider two cases. If $c_{n-k} = 0$ we recall that c_{n-k} equals the second digit of $\alpha' + 1$ so that $\alpha' = B - 1$ and $\upsilon_{n-k} = 1$ (a carry is necessary). In the case $c_{n-k} > 0$, however, we can conclude that $\alpha' = c_{n-k} - 1$.

Step 3*: What about* α*?* Suppose that $c_{k+1} = \alpha'$. This implies that $\alpha = \upsilon_{k+1}$. And what happens if $\alpha' \neq c_{k+1}$? If $c_{k+1} = 0$ this is possible only if $\alpha' = B - 1$ and then necessarily $\alpha = 0$ (since $R_{k+1} < 2B - 1$). In the case $c_{k+1} > 0$ we can conclude that the carry, if there is one, was generated by α, i.e., $\alpha = \upsilon_{k+1}$.

Step 4*:* υ_{n-k} *again.* For certain cases υ_{n-k} was calculated already in Step 2. But now we know more: υ_{n-k} can be determined easily from α, α' and υ_{n-k+1}: if $\alpha = 1$ or $\alpha' + \upsilon_{n-k+1} = B$ then $\upsilon_{n-k} = 1$, and otherwise it follows that $\upsilon_{n-k} = 0$. □

The *proof of the theorem* is now easy: Write $n = 2r$ or $n = 2r + 1$. By Lemma 2 there are F_{2r} elements in $T_{B,n}$, and by Lemma 3 there is a bijection between $T_{B,n}$ and the range of $\phi_{B,n}$.

We conclude this note with some **examples and remarks:**

1. The numbers R_k lie in $\{2B-2, B, B-1, B-2, 0\}$, and the kth digit of the final result is the last digit of $R_k + \upsilon_{k+1}$. This explains why all digits of the $\phi_{B,n}(a)$ lie in $\{0, 1, B-1, B-2\}$.

2. If one deals with 4 digits there will be $F_4 = 3$ different numbers in the range of $\phi_{B,4}$. In the following table they are depicted for the case $B = 10$, and the associated $t_1t_2t_3t_4$ are also shown. For example, all $a = (a_1a_2a_3a_4)_{10} \in I^*_{10,4}$ for which the associated $t_1t_2t_3t_4$ equals 0101 (i.e., all a with $a_2 < a_3$) give rise to $\phi_{4,10}(a) = 9999$.

$t_1t_2t_3t_4$	0101 (or: $a_2 < a_3$)	0011 (or: $a_2 > a_3$)	0111 (or: $a_2 = a_3$)
	9999	10890	10989

3. And here are the three numbers for $n = 5$:

$t_1t_2t_3t_4t_5$	00111 (or: $a_2 < a_4$)	01001 (or: $a_2 > a_4$)	01111 (or: $a_2 = a_4$)
	99099	109890	109989

4. The preceding tables can easily be transformed for the case of B-expansions: $1, 0, 8, 9$ have to be replaced by $1, 0, B-2, B-1$, respectively.

5. Here are two proposals how to use the results of this paper for a *mathematical magical trick*:

a) Let a spectator choose a number $a = a_1a_2a_3a_4$ with 4 digits. (Unless one works with mathematicians one should use the decimal system.) Two conditions should be satisfied: $a_1 > a_4$ and $a_2 > a_3$.

   Then let him or her calculate $\phi_{10,4}(a)$. You, the magician, have prepared an envelope with the prediction $\phi_{10,4}(a) = 10890$ and you can be sure that it will be true. If you prefer to impose the condition $a_2 < a_3$ then the result of the spectator's calculation will be 9999.

b) The same idea can be used for integers of arbitrary length. We illustrate this idea for numbers with 10 digits in the decimal system:

   Your spectator chooses 5 pairs of digits: $(x_1, y_1), \ldots, (x_5, y_5)$ (with x_i, y_i in the set $\{0, 1, \ldots, 9\}$). The only condition is that $x_k > y_k$ for all k. From these pairs we glue together the number $a = x_1x_2 \ldots x_ny_n \ldots y_1$ with 10 digits, i.e., we put together first all x_k and then the y_k in reverse order. Then we are sure that $t_1 \ldots t_{10} = 0000011111$, and thus we can predict the result by calculating $\phi_{10,10}(a)$ for any a of this type: one can guarantee that we will arrive at 10999890000.

6. Readers who are interested in another connection between Fibonacci numbers and mathematical magic should consult the paper "Fibonacci goes magic" by the author of this note (Elemente der Mathematik 68, 2013, pp. 1–9).

Acknowledgement
The author wishes to express his gratitude to the referee for many helpful remarks.

Ehrhard Behrends
Mathematisches Institut
Freie Universität Berlin
Arnimallee 6
D-14 195 Berlin, Germany
e-mail: behrends@math.fu-berlin.de