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On a family of pseudohyperbolic disks

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1 Introduction

Hyperbolic geometry [1] *was created in the first half of the nineteenth century in the midst of attempts to understand Euclid's axiomatic basis for geometry.* The mathematicians at that time were mainly driven by the question whether the parallel axiom was redundant or not. It turned out that it was not. *Hyperbolic geometry is now one type of non-Euclidean geometry that discards the parallel axiom. Einstein and Minkowski found in non-Euclidean geometry a geometric basis for the understanding of physical time and space. These negatively curved geometries, of which hyperbolic non-Euclidean geometry is the prototype, are the generic forms of geometry. They have profound applications to the study of com-*

[1] The italic text stems from [3]

In der Funktionentheorie der Einheitskreisscheibe $\mathbb{D}$ spielt die hyperbolische Geometrie eine zentrale Rolle. Bekanntlich sind wegen des Lemmas von Schwarz–Pick die holomorphen Isometrien bezüglich dieser Geometrie nichts anderes als die konformen Selbstabbildungen von $\mathbb{D}$. Über das Konvergenzverhalten einer Potenzreihe am Rand ihres Konvergenzkreises gibt der Abelsche Grenzwertsatz Auskunft. Dabei spielt der sogenannte Stolz-Winkel eine zentrale Rolle. In der vorliegenden Arbeit untersuchen die Autoren, ob die Schar der Kreisscheiben $D_\rho(x,r)$ mit festem Radius r und $-1 < x < 1$ bezüglich der pseudohyperbolischen Metrik ρ in $\mathbb{D}$ einen solchen Stolz-Winkel bilden. Dazu bestimmen sie mit Hilfe funktionentheoretischer Mittel explizit die Einhüllende der besagten Kreisschar.

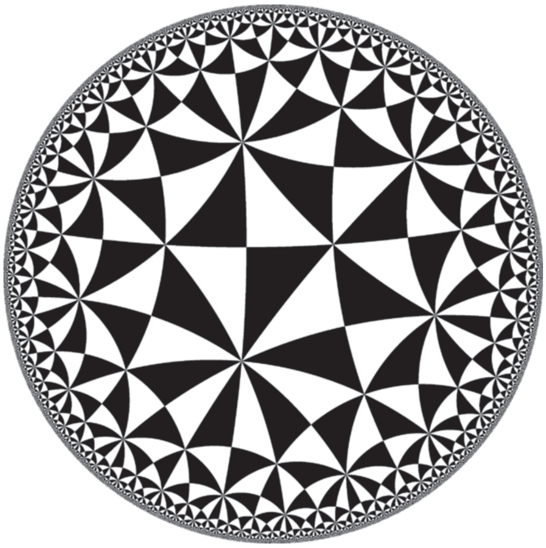

Fig. 1 Tilings of the Poincaré disk [6]

plex variables, to the topology of two- and three-dimensional manifolds, to group theory, to physics, and to other disparate fields of mathematics. Outside mathematics, hyperbolic tesselations of the unit disk have been rendered very popular by the artist M.C. Escher. A nice introduction into hyperbolic geometry is, for example, given in the monograph [1] and in [3].

In our note we are interested in the Poincaré disk model. So let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the (complex) plane which we identify with $\mathbb{R}^2$. The lines/geodesics with respect to the hyperbolic geometry in this model are arcs of Euclidean circles in $\mathbb{D}$ that are orthogonal to the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{D}$ (see Figure 2). Given a line C in the hyperbolic geometry and a point $a \in \mathbb{D}$ not belonging to C, there are infinitely many hyperbolic lines parallel to C (in other words disjoint from C) and passing through a (see Figure 2). The hyperbolic distance $P(a, b)$ of two points a and b is the hyperbolic length of the associated geodesic and is therefore given by the integral $L(\gamma) := \int_\gamma \frac{2\,|dz|}{1-|z|^2}$ over the unique circular arc γ passing through a and b and orthogonal to $\mathbb{T}$. Note that $L(\gamma) = \inf L(\Gamma)$, where Γ is any smooth curve joining a with b. Or if one prefers a nice formula:

$$\frac{|a-b|^2}{(1-|a|^2)(1-|b|^2)} = \frac{1}{2}\left(\frac{e^{P(a,b)} + e^{-P(a,b)}}{2} - 1\right).$$

Let

$$\rho(a, b) = \tanh\left(\frac{1}{2}P(a, b)\right).$$

Then $\rho(a, b)$ is called the pseudohyperbolic distance of the two points a, b and is given by

$$\rho(a, b) := \left|\frac{a-b}{1-\overline{a}b}\right|.$$

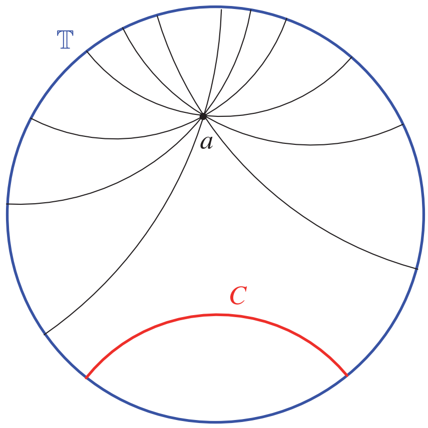

Fig. 2 Infinitely many lines parallel to line C and passing through point a.

In other words,

$$P(a, b) = \log \frac{1 + \rho(a, b)}{1 - \rho(a, b)}.$$

It is this pseudohyperbolic distance that we will work with, because this seems to be the most suitable for function theoretic aspects.

2 Function theoretic tools

Given $a \in \mathbb{D}$ and $0 < r < 1$, let

$$D_\rho(a, r) = \{z \in \mathbb{D} : \rho(z, a) < r\}$$

be the pseudohyperbolic disk centered at a and with radius r. It is a simple computational exercise in complex analysis, that $D_\rho(a, r)$ coincides with the Euclidean disk $D(p, R)$ where

$$p = \frac{1 - r^2}{1 - r^2|a|^2}\, a \text{ and } R = \frac{1 - |a|^2}{1 - r^2|a|^2}\, r.$$

An important feature of the hyperbolic metric within function theory comes from the Schwarz–Pick lemma which tells us that the holomorphic isometries with respect to ρ (or P) are exactly the conformal self-mappings of the disk:

Theorem 2.1 (Schwarz–Pick Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then, for every $z, w \in \mathbb{D}$,*

$$\rho(f(z), f(w)) \leq \rho(z, w),$$

with equality at a pair (z, w), $z \neq w$, if and only if

$$f(z) = e^{i\theta} \frac{a - z}{1 - \overline{a}z}$$

for some $a \in \mathbb{D}$ and $\theta \in [0, 2\pi[$.

Proof. This is an immediate corollary to the Schwarz lemma (see, e.g., [5]) by considering the function

$$F := S_{f(w)} \circ f \circ S_w,$$

where for $a \in \mathbb{D}$,

$$S_a(z) = \frac{a - z}{1 - \overline{a}z}$$

is the conformal automorphism of $\mathbb{D}$ interchanging a with the origin. $\square$

Consider now the set of all pseudohyperbolic disks $D_\rho(x, r)$, $x \in]-1, 1[$, with fixed radius $r \in]0, 1[$. In studying the boundary behaviour of holomorphic functions in the disk, it is of interest to know whether the set $\bigcup_{x\in]-1,1[} D_\rho(x, r)$ belongs to a cone

$$\Delta(\beta) := \left\{ z \in \mathbb{D} : \frac{|\mathrm{Im}\, z|}{1 - \mathrm{Re}\, z} < \tan \beta \right\}$$

with cusp at $z = 1$ and angle 2β such that $0 < \beta < \pi/2$. A positive answer is known among specialists in hyperbolic geometry. We never encountered a proof, though, available for function theorists. It is the aim of this note to provide such a proof. For a nice introduction into the function theoretic aspects of the hyperbolic geometry, see [2].

3 A union of hyperbolic disks

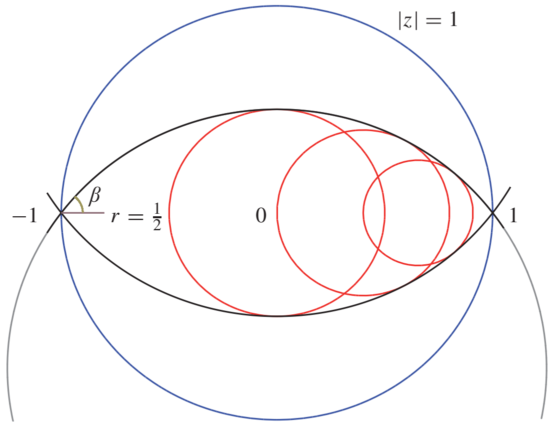

Fig. 3 The boundary of a union of hyperbolic disks with fixed radius

Here is the assertion we are going to prove.

Theorem 3.1.

(1) *The upper boundary* $\mathscr{C}^+$ *of* $\bigcup_{-1<x<1} D_\rho(x, r)$ *is an arc of the circle*

$$\mathfrak{C} := \left\{ w \in \mathbb{C} : \left| w + i\frac{1 - r^2}{2r} \right| = \frac{1 + r^2}{2r} \right\},$$

the lower boundary is its reflection with respect to the real axis (*see Figure* 3).

(2) *The tangens of the angle β under which $\mathscr{C}^+$ cuts the real axis is* $2r/(1-r^2)$.

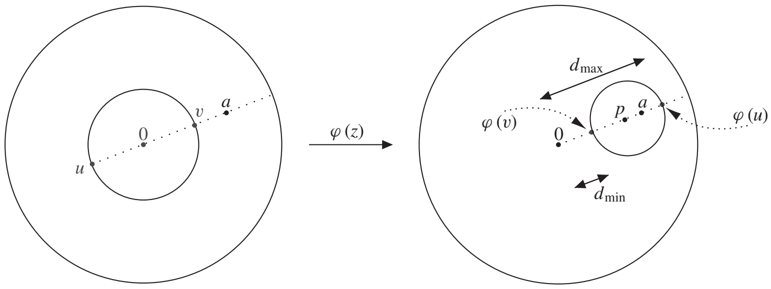

Fig. 4 Hyperbolic disks

Before we give our proof, we observe that the largest distance $d_{\max}$, respectively, if $|a| > r$, the smallest distance $d_{\min}$, of a point in $D_\rho(a, r)$ to 0 are given by

$$d_{\min} = \frac{|a| - r}{1 - r|a|} \quad \text{and} \quad d_{\max} = \frac{|a| + r}{1 + r|a|}.$$

This can be seen by considering the conformal automorphism of the disk given by $\varphi(z) = \frac{a-z}{1-\overline{a}z}$, by noticing that the image of the disk $D(0, r) = D_\rho(0, r)$ is the disk $D(p, R)$ and by calculating the images of the boundary points $\pm re^{i \arg a}$ which lie on the half-line passing through 0 and a (see Figure 4).

Proof. (1) The proof is best done via a conformal mapping of $\mathbb{D}$ onto the right half-plane (see Figure 5).

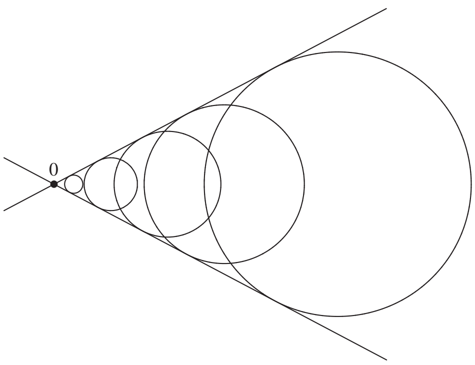

Fig. 5 The boundary of a union of hyperbolic disks with fixed radius in the right half-plane

Recall that if $-1 < x < 1$, then the function S_x, given by $S_x(z) = (x - z)/(1 - xz)$, maps the disk $D(0, r)$ onto $D_\rho(x, r)$ with $x_M := S_x(-r) = (x + r)/(1 + xr)$ and $x_m := S_x(r) = (x - r)/(1 - xr)$. Since S_x maps $[-1, 1]$ onto $[-1, 1]$, and since the circle $D(0, r)$ cuts $[-1, 1]$ at a right angle, the angle invariance property of conformal maps implies that

$D_\rho(x,r)$ is the disk passing through the points x_m and x_M and orthogonal to $[-1,1]$. Now we switch to the right half-plane H by using the map $\Psi(z) := (1+z)/(1-z)$ of $\mathbb{D}$ onto H. Then, by a similar reasoning, $K := \Psi(D_\rho(x,r))$ is the disk orthogonal to the real axis and passing through the points

$$w_m := \Psi(x_m) = \frac{1-r}{1+r}\,\frac{1+x}{1-x} \quad \text{and} \quad w_M := \Psi(x_M) = \frac{1+r}{1-r}\,\frac{1+x}{1-x}.$$

Hence the center C_x of K is the arithmetic mean $(w_M + w_m)/2$ of w_m and w_M, and the radius R_x is $(w_M - w_m)/2$. Thus

$$C_x = \frac{1+r^2}{1-r^2}\,\frac{1+x}{1-x} \quad \text{and} \quad R_x = \frac{2r}{1-r^2}\,\frac{1+x}{1-x}.$$

Note that if the center x of the pseudohyperbolic disk $D_\rho(x,r)$ runs through $]-1,1[$, then the center C_x of the Euclidean disk $\Psi(D_\rho(x,r))$ runs through $]0,\infty[$. Due to conformal invariance, the boundary $\mathscr{C}$ of $\bigcup_{-1<x<1} D_\rho(x,r)$ coincides with the preimage $\Psi^{-1}(\widetilde{\mathscr{C}})$ of the boundary $\widetilde{\mathscr{C}}$ of

$$S := \bigcup_{-1<x<1} \Psi(D_\rho(x,r)) = \bigcup_{-1<x<1} K(C_x, R_x).$$

We show that $\widetilde{\mathscr{C}}$ is the union of the half-line $\{e^{i\beta}t : t \geq 0\}$ and its reflection $\{e^{-i\beta}t : t \geq 0\}$, where β is the angle with $\sin\beta = 2r/(1+r^2)$ (for a first glimpse, see Figure 5).

For a fixed $x \in\,]-1,1[$, consider in the first quadrant the tangent T to $K(C_x, R_x)$ that passes through the origin. Let β_x be its angle with respect to the real axis. Then

$$\sin\beta_x = \frac{R_x}{C_x} = \frac{2r}{1+r^2}.$$

This is independent of x. Hence T is a joint tangent to all the Euclidean disks $K(C_x, R_x)$. In other words, S is contained in the infinite triangle Δ formed by T and its reflection. To show that $\Delta = S$, we need to prove that every point on T is the tangent point of some of the disks $K(C_a, R_a)$ with $-1 < a < 1$. To this end, let P be the point on T whose distance to 0 is t and let T_t be the line orthogonal to T and passing through P. Then T_t cuts the real line at a point x_t. The unique disk K centered at x_t and having P as its tangent point to T has center $C(t)$ and radius $R(t)$, which are given by

$$C(t) = x_t \quad \text{and} \quad R(t) = x_t \sin\beta = x_t\ 2r/(1+r^2)$$

(see Figure 6). Now

$$x_t = \frac{t}{\cos\beta} = \frac{t}{\sqrt{1-\sin^2\beta}} = t\,\frac{1+r^2}{1-r^2}.$$

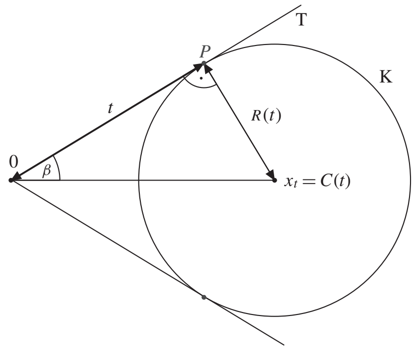

Fig. 6 The tangent T

But $t = (1+a)(1-a)$ for a unique $a \in \,]-1, 1[$. Thus

$$x_t = \frac{1+a}{1-a}\frac{1+r^2}{1-r^2} = C_a$$

and

$$R(t) = x_t \; 2r/(1+r^2) = \frac{1+a}{1-a}\frac{2r}{1-r^2} = R_a.$$

We conclude that

$$K = \Psi(D_\rho(a,r)) = K(C_a, R_a).$$

By moving back from the right half-plane to the unit disk, we see that $\mathscr{C}^+ = \Psi^{-1}(T)$ is an arc of a circle $\mathfrak{C}$ which passes through -1 and 1 and cuts twice the axis $[-1, 1]$ under the angle β with $\sin\beta = 2r/(1+r^2)$. Using Figure 7, we then deduce that the radius R of $\mathfrak{C}$ coincides with the hypotenuse of the displayed triangle and so

$$R = \frac{1}{\cos\alpha} = \frac{1}{\sin(\frac{\pi}{2}-\alpha)} = \frac{1}{\sin\beta} = \frac{1+r^2}{2r}.$$

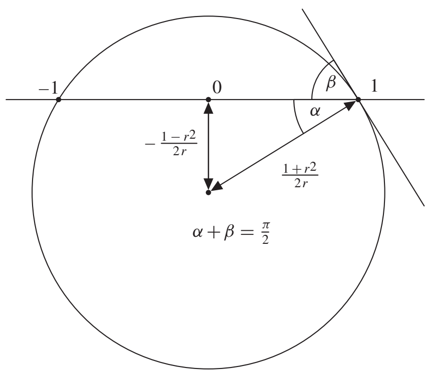

Fig. 7 The angle β

This implies that the center C of $\mathfrak{C}$ is given by

$$C = -i\,R\sin\alpha = -i\,\frac{1+r^2}{2r}\sqrt{1-\cos^2\alpha} = -i\,\frac{1+r^2}{2r}\,\frac{1-r^2}{1+r^2} = -i\,\frac{1-r^2}{2r}.$$

(2) $\tan\beta = \sin(\pi/2-\alpha)/\cos(\pi/2-\alpha) = \cos\alpha/\sin\alpha$ with

$$\sin\alpha = \frac{\frac{1-r^2}{2r}}{\frac{1+r^2}{2r}} = \frac{1-r^2}{1+r^2} \text{ and } \cos\alpha = \frac{1}{\frac{1+r^2}{2r}} = \frac{2r}{1+r^2}.$$

Hence $\tan\beta = 2r/(1-r^2)$ (see Figure 7). □

A purely computational proof can be found in [4, Appendix]. There it is also shown that the Euclidean length of $\mathscr{C}^+$ is $2\frac{1+r^2}{r}\arctan r$, and that the surface enclosed by $\bigcup_{-1<x<1} D_\rho(x,r)$ has Euclidean measure $\left(\frac{1+r^2}{r}\right)^2 \arctan r - \frac{1-r^2}{r}$.

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