
Pascal, Fibonacci, and geometry

Jean Pedersen and Hans Walser

Jean Pedersen (1934–2016) was a Professor of Mathematics in the Department of Mathematics and Computer Science at Santa Clara University. Her principal research involved studying the interconnections between systematic paper folding, polyhedral geometry, combinatorics, and number theory.

Hans R. Walser is a now retired college teacher. He was also teaching at the ETH-Zürich (Federal institute of technology, Zürich) and at Basel University. His main interest is the didactics of geometry, especially geometry in space, polyhedra and the Golden Section.

1 Background

The fact that the Fibonacci numbers appear by adding up appropriate entries in slanting rows of Pascal's triangle is by now almost ubiquitous. The purpose of this article is to show how the authors eventually came to view the Fibonacci numbers as they relate to the Pascal triangle so that the results may be generalized.

2 Notation

In what follows we will use an expanded notation for the binomial coefficient $\binom{n}{r}$ that renders it amenable to a more symmetric geometric interpretation and easier to generalize.

Thus we will write

$$\binom{n}{r} = \binom{n}{r \quad s} = \frac{n!}{r!s!}, \text{ where } r + s = n \geq 0. \quad (1)$$

Das Pascalsche Dreieck der Binomialkoeffizienten ist ein beliebtes Tummelfeld für zahlentheoretische Entdeckungen im Unterricht. Insbesondere erscheinen die Fibonacci-Zahlen als Schrägzeilensummen. Diese Ideen lassen sich in den Raum übertragen. Anstelle des Dreiecks erhalten wir die Pascal Pyramide der Trinomialkoeffizienten. Statt mit Schrägzeilen wird dann entsprechend mit schrägen Schnittebenen durch die Pascal Pyramide gearbeitet, um die Tribonacci Zahlen zu erhalten.

Further, we define

$$\binom{n}{r \ s} = 0, \text{ if either } r < 0 \text{ or } s < 0. \quad (2)$$

The Pascal identity, usually written as $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$, now becomes

$$\binom{n-1}{r-1 \ s} + \binom{n-1}{r \ s-1} = \binom{n}{r \ s}. \quad (3)$$

Notice how easy it is, with this notation, to guess what the corresponding identity would be for trinomial coefficients $\binom{n}{r \ s \ t}$.

3 The Fibonacci numbers

We now place the binomial coefficients $\binom{n}{r \ s}$ in a hexagonal lattice as shown in Figure 1. Two of the zero entries have been included, since they pertain to a particular, but not special case, that illustrates why the sum of the numbers along slanting rows gives the Fibonacci numbers.

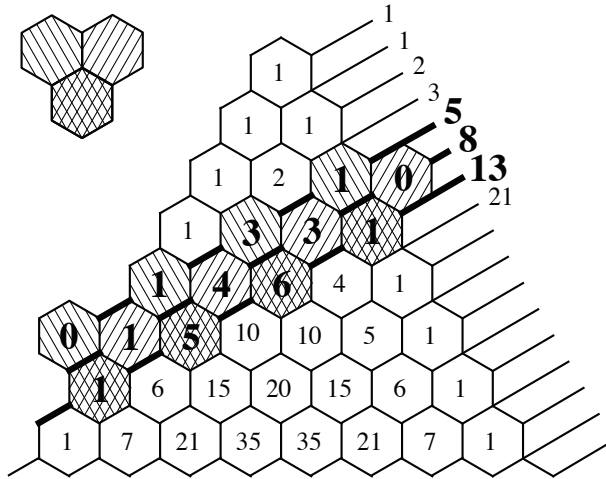


Fig. 1 Visualization of the Fibonacci recurrence.

The sum of a number in the upper slanting row and a number in the middle row is, by the Pascal identity (3), a number of the lower slanting row, which is crosswise shaded. Furthermore, every number in the upper slanting row, and every number in the middle row, is used exactly once as part of a number in the lower slanting row. Hence the sums in the slanting rows satisfy the Fibonacci recurrence relation

$$F_k = F_{k-1} + F_{k-2}. \quad (4)$$

Since $F_1 = F_2 = 1$ we get the usual Fibonacci sequence.

4 The geometry of slanting lines

We use the affine coordinate system of Figure 2.

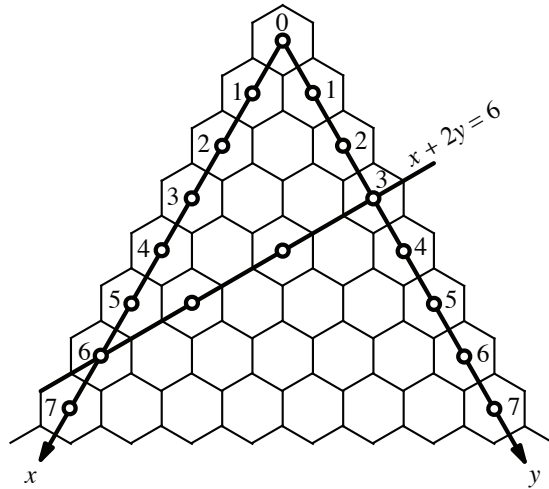


Fig. 2 Coordinate system.

In this coordinate system we consider the slanting lines of Figure 1 that have the equation:

$$x + 2y = k, \text{ where } k \geq 0. \tag{5}$$

The centers of the hexagons have non-negative integer coordinates. Hence we get a new way to look at the Fibonacci numbers.

As illustration consider the particular but not special case $k = 6$ shown in Figure 2. The corresponding line has the equation $x + 2y = 6$ with the following non-negative integer solutions: $(0, 3), (2, 2), (4, 1), (6, 0)$. Now we take the sum S_6 of the corresponding binomial coefficients of the form $\binom{x+y}{y \ x}$, with $x + 2y = 6$, (going down the sloping line from right to left)

$$S_6 = \binom{3}{3 \ 0} + \binom{4}{2 \ 2} + \binom{5}{1 \ 4} + \binom{6}{0 \ 6} = 1 + 6 + 5 + 1 = 13,$$

and find that the sum S_6 is the Fibonacci number F_7 . In general we have:

$$S_k = \sum_{\substack{\text{non-negative} \\ \text{integer solutions} \\ \text{of } x+2y=k}} \binom{x+y}{y \ x} = F_{k+1}. \tag{6}$$

5 The trinomial coefficients and the Pascal pyramid

We deal with the trinomial coefficients

$$\binom{n}{r \ s \ t} = \frac{n!}{r!s!t!}, \text{ where } r + s + t = n \tag{7}$$

with the Pascal identity

$$\binom{n-1}{r-1 \ s \ t} + \binom{n-1}{r \ s-1 \ t} + \binom{n-1}{r \ s \ t-1} = \binom{n}{r \ s \ t}. \quad (8)$$

We replace the hexagonal lattice by a spatial lattice of regular rhombic dodecahedra. Instead of a regular triangle we have now a regular tetrahedron, sometimes called the *Pascal pyramid*, shown in Figure 3.

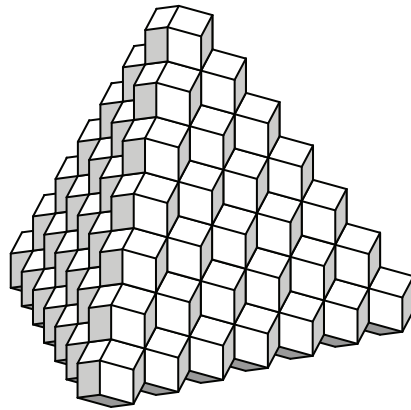


Fig. 3 The Pascal pyramid.

Again we introduce an affine coordinate system as shown in Figure 4.

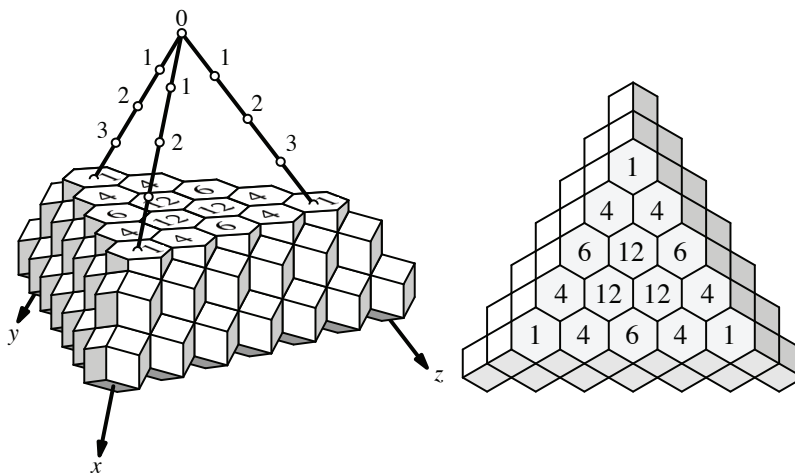


Fig. 4 Fourth layer.

Cutting the pyramid by horizontal planes we see how the trinomial coefficients sit in the horizontal layers of the Pascal pyramid (see [1, 2]). Figure 4 shows the situation in the fourth layer, the trinomial coefficients for $\binom{4}{r \ s \ t}$, where $r + s + t = 4$.

6 Slanting planes

Now we intersect the Pascal pyramid with slanting planes p_k with the equations:

$$p_k : x + 2y + 3z = k, \text{ where } k \geq 0.$$

Figure 5 gives the situation for $k = 5$ with the equation $x + 2y + 3z = 5$. The non-negative integer solutions to this equation are $(0, 1, 1)$, $(1, 2, 0)$, $(2, 0, 1)$, $(3, 1, 0)$, $(5, 0, 0)$.

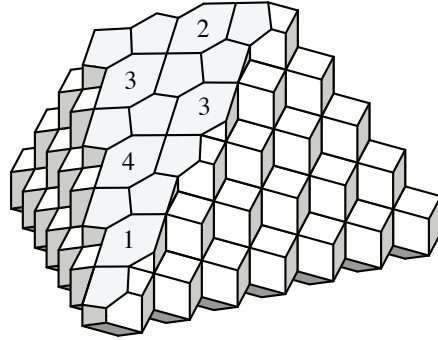


Fig. 5 $k = 5$.

The sum T_5 of all trinomial coefficients of the form $\binom{x+y+z}{x \ y \ z}$, with $x + 2y + 3z = 5$, in the plane p_5 is:

$$\begin{aligned} T_5 &= \binom{2}{0 \ 1 \ 1} + \binom{3}{1 \ 2 \ 0} + \binom{3}{2 \ 0 \ 1} + \binom{4}{3 \ 1 \ 0} + \binom{5}{5 \ 0 \ 0} \\ &= 2 + 3 + 3 + 4 + 1 = 13. \end{aligned}$$

Now we define in general T_k the sum of all trinomial coefficients of the plane p_k . For T_k we have the recurrence relation:

$$T_k = T_{k-1} + T_{k-2} + T_{k-3}. \quad (9)$$

Proof. Let (r, s, t) be a point of p_k , i.e., a solution of $x + 2y + 3z = k$. Then $(r - 1, s, t)$ is a solution of $x + 2y + 3z = k - 1$, hence a point of p_{k-1} , and $(r, s - 1, t)$ a solution of $x + 2y + 3z = k - 2$, a point of p_{k-2} , and finally $(r, s, t - 1)$ a solution of $x + 2y + 3z = k - 3$, i.e., a point of p_{k-3} .

Because of the Pascal identity (8) every summand of T_k is the sum of a particular summand of T_{k-1} , T_{k-2} , and T_{k-3} . On the other side, every summand of T_{k-1} , T_{k-2} , or T_{k-3} appears exactly once in T_k .

A rhombic dodecahedron with its center on the plane p_k is touched on its "roof" with the three rhombic dodecahedra whose centers lay on the three parallel planes p_{k-1} , p_{k-2} , and p_{k-3} respectively.

For the starting values T_0, T_1, T_2 we get respectively:

$k = 0$: The equation $x + 2y + 3z = 0$ has the only non-negative integer solution $(0, 0, 0)$. Since $\binom{0}{0 \ 0} = 1$ we have $T_0 = 1$.

$k = 1$: The equation $x + 2y + 3z = 1$ has the only non-negative integer solution $(1, 0, 0)$. Therefore we have $T_1 = \binom{1}{1 \ 0} = 1$.

$k = 2$: The equation $x + 2y + 3z = 2$ has the non-negative integer solutions $(0, 1, 0)$ and $(2, 0, 0)$. Therefore we have $T_2 = \binom{1}{0 \ 1} + \binom{2}{2 \ 0} = 1 + 1 = 2$.

The general formula for T_k is:

$$T_k = \sum_{\substack{\text{non-negative} \\ \text{integer solutions} \\ \text{of } x+2y+3z=k}} \binom{x+y+z}{x \ y \ z}.$$

From the starting values 1, 1, and 2 the recurrence relation (9) yields the sequence of the so-called *Tribonacci numbers*:

$$1, 1, 2, 4, 7, 13, 24, 44, \dots \quad \square$$

References

- [1] P. Hilton, D. Holton, and J. Pedersen (2002): *Mathematical Vistas: From a room with many windows*, Springer, NY.
- [2] P. Hilton and J. Pedersen (2012): *Mathematics, Models, and Magz, Part 1: Investigating patterns in Pascal's triangle and tetrahedron*, *Math. Mag.*, 85, 97–109.

Jean Pedersen

Hans Walser
 Department of Mathematics
 Basel University
 CH-4051 Basel, Switzerland
 e-mail: hwalser@bluewin.ch