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Elemente der Mathematik

Sinc integrals and tiny numbers

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Von den Borwein-Integralen

$$B_n := \int_0^\infty \prod_{k=1}^{n} \operatorname{sinc}(a_k x)\,\mathrm{d}x, \quad a_k = \frac{1}{2k-1},$$

ist bekannt, dass $B_1 = B_2 = \cdots = B_7 = \frac{\pi}{2}$ aber $B_n < \frac{\pi}{2}$ für $n \geq 8$. Die Abweichung vom Wert $\frac{\pi}{2}$ ist bei $n = 8$ enorm klein. Betrachtet man für $\lambda \geq 1$ die Variante

$$I(\lambda) = \lambda \int_0^\infty \operatorname{sinc}(\lambda x) \prod_{k=1}^{n} \operatorname{sinc}(a_k x)\,\mathrm{d}x,$$

wobei n so gewählt wird, dass $\sum_{k=1}^{n} a_k > \lambda \geq \sum_{k=1}^{n-1} a_k$, so folgt aus einem Resultat von David und Jon Borwein

$$I(\lambda) = \frac{\pi}{2}\,(1 - t(\lambda)) \quad \text{mit} \quad t(\lambda) = \frac{(a_1 + a_2 + \cdots + a_n - \lambda)^n}{2^{n-1}\,n!\,\prod_{k=1}^{n} a_k}\,.$$

Mit wachsendem λ werden die Zahlen $t(\lambda)$ schnell sehr klein. In der vorliegenden Arbeit wird gezeigt, wie numerische Werte dieser Zahlen, und damit die Integrale $I(\lambda)$, schnell und mit hoher Genauigkeit berechnet werden können. Zum Beispiel ist der Integrand von $I(10)$ das Produkt von 68100152 sinc-Funktionen und

$$t(10) = 9.6492736004286844634795531209398105309232\ldots \cdot 10^{-554381308}\,.$$

1 Introduction

The sinc function is defined as

$$\operatorname{sinc}(x) = \begin{cases} \sin(x)/x & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$$

We consider the integral

$$I_n(\lambda) = \lambda \int_0^\infty \operatorname{sinc}(\lambda x) \prod_{k=1}^{n} \operatorname{sinc}\left(\frac{x}{2k-1}\right) \mathrm{d}x\,, \tag{1}$$

where λ is a real number ≥ 1. For abbreviation we put

$$s(n) := \sum_{k=1}^{n} \frac{1}{2k-1}\,. \tag{2}$$

From [1, pp. 3–4] it is known that $I_n(\lambda) = \pi/2$ if $s(n) \leq \lambda$, and $I_n(\lambda) < \pi/2$ if $s(n) > \lambda$. We will show that, when n becomes large enough that

$$s(n) > \lambda \geq s(n-1) \tag{3}$$

holds, then $I_n(\lambda)$ can be less than $\pi/2$ by a *very* tiny amount.

From Corollary 1 of [4] (see also Theorem 2 of [3]), we know that, if

$$2a_k \geq a_n > 0 \quad \text{for} \quad k = 0, 1, \ldots, n-1, \tag{4}$$

and

$$\sum_{k=1}^{n} a_k > a_0 \geq \sum_{k=1}^{n-1} a_k\,, \tag{5}$$

then

$$\int_0^\infty \prod_{k=0}^{n} \frac{\sin(a_k x)}{x} \,\mathrm{d}x = \frac{\pi}{2}\left(\prod_{k=1}^{n} a_k - \frac{(a_1 + a_2 + \cdots + a_n - a_0)^n}{2^{n-1}\, n!}\right). \tag{6}$$

Eq. (6) may be written as

$$a_0 \int_0^\infty \prod_{k=0}^{n} \operatorname{sinc}(a_k x) \,\mathrm{d}x = \frac{\pi}{2}\left(1 - \frac{(a_1 + a_2 + \cdots + a_n - a_0)^n}{2^{n-1}\, n!\, \prod_{k=1}^{n} a_k}\right). \tag{7}$$

We put

$$a_0 = \lambda\,, \qquad a_k = \frac{1}{2k-1} \quad \text{for} \quad k = 1, 2, \ldots, n, \tag{8}$$

so the left-hand side of Eq. (7) is equal to $I_n(\lambda)$ (see Eq. (1)). Since $\lambda \geq 1$, the a_k in Eq. (8) satisfy the inequalities (4): they are all positive with $a_k \geq a_n$ for $k = 0, 1, \ldots, n-1$, which implies that $2a_k \geq a_n$. If, in addition, (5) and, equivalently, (3) hold, then the value of $I_n(\lambda)$ is given by the right-hand side of (7).

If inequalities (3) hold, then n is uniquely determined by the value of λ. In this case, we can use the simpler notation $I(\lambda)$ instead of $I_n(\lambda)$. We have

$$\prod_{k=1}^{n} a_k = \frac{1}{2n-1} \cdot \frac{1}{2n-3} \cdots \frac{1}{3} \cdot \frac{1}{1} = \frac{2n \cdot (2n-2) \cdots 4 \cdot 2}{2n \cdot (2n-1) \cdots 2 \cdot 1} = \frac{2^n n!}{(2n)!},$$

so it easily follows that

$$I(\lambda) = \lambda \int_0^\infty \operatorname{sinc}(\lambda x) \prod_{k=1}^{n} \operatorname{sinc}\left(\frac{x}{2k-1}\right) \mathrm{d}x = \frac{\pi}{2}\,(1 - t(\lambda)), \tag{9}$$

where

$$t(\lambda) = \frac{(s(n) - \lambda)^n}{2^{2n-1}} \cdot \frac{(2n)!}{n!^2}. \tag{10}$$

We now describe the problem to be solved in the present paper: $t(\lambda)$ can be a *very* tiny number. *We will show how to calculate these numbers, and therefore the integrals $I(\lambda)$, with high precision in short time.* Numerical examples will be given for integer values of λ.

2 Calculating $t(\lambda)$ for $\lambda = 1, \ldots, 10$

Given the value of λ, the first task is to find the value of n such that the inequalities (3) are satisfied. For $\lambda \leq 10$, it is not difficult to find this by simply computing the partial sums $s(n)$ until one finally exceeds λ. Then, with the help of (10) we compute the decimal approximations of the $t(\lambda)$ values for $\lambda = 1, 2, \ldots, 10$, shown in Table 1. These values are rounded in the last (40th) decimal place. In *Mathematica* on a standard laptop, only the last two n values took more than a minute to calculate. One sees that the numbers $t(\lambda)$ quickly become rather tiny.

Examples: For $\lambda = 1$ we have

$$s(1) = 1 \leq \lambda < 1 + \frac{1}{3} = \frac{4}{3} = s(2)$$

and therefore, using the inequalities (3), $n = 2$. Hence Eq. (9) delivers

$$t(1) = \frac{(\frac{4}{3} - 1)^2}{2^3} \frac{4!}{2!^2} = \frac{1}{12} \qquad \text{and} \qquad I(1) = \frac{\pi}{2}\left(1 - \frac{1}{12}\right) = \frac{11\pi}{24} \approx 0.458333\,\pi\,.$$

Mathematica can calculate this integral directly:

$$I(1) = \int_0^\infty \operatorname{sinc}(1 \cdot x)\ \operatorname{sinc}\left(\frac{x}{1}\right)\ \operatorname{sinc}\left(\frac{x}{3}\right) \mathrm{d}x = \frac{11\,\pi}{24}\,.$$

For $\lambda = 2$ we find

$$s(7) = \frac{88069}{45045} = 1.95513\ldots < \lambda < 2.02181\ldots = \frac{91072}{45045} = s(8)\,,$$

hence $n = 8$, and, therefore,

$$t(2) = \frac{(\frac{91072}{45045} - 2)^8}{2^{15}} \frac{16!}{8!^2} = \frac{3377940044732998170721}{168579263752214678679209808915000000},$$

$$I(2) = \frac{\pi}{2}(1 - t(2)) = \frac{168579263752211300739165075916829279\,\pi}{337158527504429357358419617830000000}$$
$$\approx 0.49999999999998998115\,\pi\,.$$

Mathematica is able to calculate

$$I(2) = 2\int_0^\infty \operatorname{sinc}(2\cdot x)\ \operatorname{sinc}\left(\frac{x}{1}\right)\ \operatorname{sinc}\left(\frac{x}{3}\right)\cdots\ \operatorname{sinc}\left(\frac{x}{15}\right)\mathrm{d}x$$

directly and finds the same result (see also [1, p. 4]).

Look what happens when we take ratios of successive n values:

$$\begin{aligned}
419/57 &\approx 7.35087719\\
3092/419 &\approx 7.37947494\\
22846/3092 &\approx 7.38874515\\
168804/22846 &\approx 7.38877703\\
1247298/168804 &\approx 7.38903107\\
9216354/1247298 &\approx 7.38905538\\
68100151/9216354 &\approx 7.38905548
\end{aligned}$$

These ratios appear to be approaching $\mathrm{e}^2 \approx 7.38905610$. That is, the n that corresponds to $\lambda+1$ is roughly e^2 times the n that corresponds to λ. Here is the explanation. The sum of N terms of the harmonic series, $\sum_{k=1}^{N} 1/k$, is about $\ln(N)$. We have $\ln(\mathrm{e}\cdot N) = \ln(N) + 1$. Therefore, if N terms of the harmonic series are required to reach a sum $S(\approx \ln(N))$, then about $\mathrm{e}\cdot N$ terms are needed to make the sum reach $S + 1$. The terms in our series $s(n)$ are about $1/2$ as large as the corresponding terms in the harmonic series. Therefore, to increase our sum by 1 requires about as many terms as the harmonic series needs to increase its sum by 2, which is about $\mathrm{e}\cdot\mathrm{e} = \mathrm{e}^2$. As a result, we can estimate that, for a given λ, the corresponding n is about $68100151\cdot \mathrm{e}^{2(\lambda-10)}$, so that n is an exponentially increasing function of λ.

Let $t(\lambda)$ be written in the form $t(\lambda) = P/Q$ with coprime integers P and Q. As λ increases, P and Q quickly become very large. For example, with $\lambda = 6$, P and Q have 453130185 and 453237210 digits, respectively. Displaying the first and last 20 digits for this case, we have

$$t(6) = \frac{P}{Q} = \frac{34293043773392420460\,(453130145\text{ digits})\,34573721229967337961}{26251415654224851611\,(453237170\text{ digits})\,00000000000000000000}\,.$$

Standard IEEE double precision ("machine precision") uses mantissas with 53 bits (about 15 decimal digits). This simply is not enough for most of the calculations we discuss here.

Eq. (10) requires that we compute the sum $s(n)$, then raise $(s(n) - \lambda)$ to a very high power. Many, or even *all*, of the significant digits in $t(\lambda)$ will be lost if we calculate $s(n)$ to only machine precision. This is caused, in part, by the well-known phenomenon that significance is lost when we subtract two numbers that are nearly equal. For example, with $\lambda = 7$, we have $n = 168804$. Computing the sum to only machine precision (in either C or *Mathematica*) gives $s(168804) \approx 7.00000179178$. We then compute $(s(168804) - 7)^{168804}$. This exponent consumes 6 significant digits, so the mantissa loses about another 6 digits.

Therefore, we did our calculations twice: first, we computed each $s(n)$ to 60 decimals, then used this value for the calculation of $t(\lambda)$ (see Eq. (10)). Then, we repeated the calculations, this time, computing each $s(n)$ to 70 decimals. These high-precision results agree with each other to more decimals than we show in Table 1. On the other hand, for $\lambda = 7$, only the first *three* digits of the machine precision calculation agree with these high-precision results. Worse, when we do the calculation for $\lambda = 8$ in machine precision, we get, approximately,

$$\frac{\pi}{2}\left(1 - 1.03496 \cdot 10^{-8742942}\right).$$

Note that *all digits and the exponent* are different from the high-precision result in Table 1.

λ	n	$t(\lambda)$
1	1	$8.33333333333333333333333333333333333333333 \cdot 10^{-2}$
2	8	$2.00376960341815534387372786892960788693394 \cdot 10^{-14}$
3	57	$4.28145410362426804246087253081144498244436 \cdot 10^{-143}$
4	419	$2.37109999756811683299146045424633182497729 \cdot 10^{-1326}$
5	3092	$2.58995444692371933547081112567031106867749 \cdot 10^{-13544}$
6	22846	$1.30633121752705800878169192300362974074011 \cdot 10^{-107025}$
7	168804	$1.20847533057118063082650540343576013360007 \cdot 10^{-970071}$
8	1247298	$6.93122229932264911357380668345493276563340 \cdot 10^{-8742945}$
9	9216354	$3.32169709990360582753676869416711999662888 \cdot 10^{-67342884}$
10	68100151	$9.64927360042868446347955312093981053092332 \cdot 10^{-554381308}$

Table 1 Values of $t(\lambda)$ for the evaluation of the integrals $I(\lambda)$

3 The program for the calculation of $t(\lambda)$ for $\lambda > 10$

When trying to calculate $t(\lambda)$ for $\lambda > 10$ one is faced with the following problems:

- n increases exponentially with λ. So, given λ, the time to find the corresponding n by computing partial sums $s(n)$ also increases exponentially.
- Therefore, the time to compute $n!$ and $(2n)!$ also grows exponentially.

- The differences $s(n) - \lambda$ become smaller as λ increases. High-precision values of $s(n) - \lambda$ are needed because these differences are raised to the power n.
- The calculations involve both very large and very small numbers. This may lead to overflow and underflow.

So our program to compute high-precision values of $t(\lambda)$ for $\lambda > 10$ in short time includes the following measures:

- We will use logarithms. Taking the natural logarithm of (10), we get

$$\ln t(\lambda) = n \ln(s(n) - \lambda) - (2n - 1)\ln 2 + \ln((2n)!) - 2\ln(n!)\,. \qquad (11)$$

- We will use the Euler–Maclaurin summation formula to calculate approximate values for $s(n)$ and the logarithms of the factorials. In order to yield the desired precision we will compute partial sums exactly to many digits and obtain error bounds.
- Let ℓ denote the value of $\lg t(\lambda) = (\ln t(\lambda))/(\ln 10)$ where $\ln t(\lambda)$ is obtained with Eq. (11) and lg is the log base 10. In order to avoid underflow when trying to evaluate $t(\lambda)$ with $t(\lambda) = 10^{\ell}$, we will extract the mantissa m and the exponent p of $t(\lambda)$, and then display $t(\lambda)$ in scientific notation

$$t(\lambda) = m \cdot 10^p \quad \text{with} \quad m = 10^{\ell - \lfloor \ell \rfloor} \quad \text{and} \quad p = \lfloor \ell \rfloor\,.$$

4 Calculating *n*

Now, we calculate the value of n that satisfies the inequalities (3) for given value of λ. With the nth harmonic number

$$H_n = \sum_{k=1}^{n} \frac{1}{k}\,,$$

Eq. (2) may be written as

$$s(n) = H_{2n-1} - \frac{1}{2} H_{n-1}\,.$$

From the inequalities (see [5, p. 76] and [7, Eq. 15])

$$\frac{1}{24(n+1)^2} < H_n - \ln\left(n + \frac{1}{2}\right) - \gamma < \frac{1}{24n^2}\,,$$

where γ is the Euler–Mascheroni constant, it follows that

$$\ln\left(2n - \frac{1}{2}\right) + \gamma + \frac{1}{24(2n)^2} < H_{2n-1} < \ln\left(2n - \frac{1}{2}\right) + \gamma + \frac{1}{24(2n-1)^2}\,,$$

$$\ln\left(n - \frac{1}{2}\right) + \gamma + \frac{1}{24n^2} < H_{n-1} < \ln\left(n - \frac{1}{2}\right) + \gamma + \frac{1}{24(n-1)^2}\,,$$

hence

$$s_{\ell}(n) < s(n) < s_u(n)\,,$$

where

$$s_\ell(n) = s^*(n) + \frac{1}{24(2n)^2} - \frac{1}{48(n-1)^2}\,, \quad s_u(n) = s^*(n) + \frac{1}{24(2n-1)^2} - \frac{1}{48n^2}\,, \tag{12}$$

with

$$s^*(n) = \ln\left(2n - \frac{1}{2}\right) - \frac{1}{2}\ln\left(n - \frac{1}{2}\right) + \frac{\gamma}{2}\,.$$

Note that

$$s_u(n) - s_\ell(n) = \frac{20n^3 - 33n^2 + 18n - 3}{96(4n^6 - 12n^5 + 13n^4 - 6n^3 + n^2)} = O(n^{-3}).$$

Therefore, if n is large, the bounds in (12) provide very good approximations to $s(n)$.

Now we consider the equation $s^*(n) = \lambda$ for a fixed value $\lambda \geq 2$. From this equation we get

$$\left(2n - \frac{1}{2}\right)\left(n - \frac{1}{2}\right)^{-1/2} = \mathrm{e}^{\lambda - \gamma/2}\,.$$

Solving for n yields

$$n = \frac{1}{8}\left(2 + \mathrm{e}^{2\lambda - \gamma} \pm \mathrm{e}^{-\gamma}\sqrt{\mathrm{e}^{4\lambda} - 4\,\mathrm{e}^{2\lambda + \gamma}}\right).$$

The equation

$$n = \left\lceil \frac{1}{8}\left(2 + \mathrm{e}^{2\lambda - \gamma} + \mathrm{e}^{-\gamma}\sqrt{\mathrm{e}^{4\lambda} - 4\,\mathrm{e}^{2\lambda + \gamma}}\right)\right\rceil, \tag{13}$$

where $\lceil\ \rceil$ denotes the ceiling function, gives an integer value of n. This n satisfies inequalities (3) if it satisfies the inequalities

$$s_u(n-1) < \lambda < s_\ell(n)\,. \tag{14}$$

Example: We will calculate n for $\lambda = 10$. This allows us to check our result against Table 1. Eq. (13) gives

$$n = \lceil 68100150.0149 \rceil = 68100151.$$

We find

$$s_u(68100150) \approx 10 - 1.09045 \cdot 10^{-10}\,,$$
$$s_\ell(68100151) \approx 10 + 7.23308 \cdot 10^{-9}\,,$$

and know that we have found the correct value of n.

5 Estimating $s(n)$

We use the Euler–Maclaurin summation formula to calculate the sum of the a_k for any large n. This method applies to general a_k, and gives us an estimate of the error.

One version of the Euler–Maclaurin summation formula is (see, e.g., [6, pp. 542–543])

$$\sum_{k=m}^{n} f(k) = \int_m^n f(x)\,\mathrm{d}x + \frac{f(m)+f(n)}{2} + \sum_{j=1}^{\mu} \frac{B_{2j}}{(2j)!}\Big(f^{(2j-1)}(n) - f^{(2j-1)}(m)\Big) + R_\mu(m,n) \tag{15}$$

with the remainder term

$$R_\mu(m,n) = \int_m^n \frac{B_{2\mu+1}(x-\lfloor x\rfloor)}{(2\mu+1)!}\, f^{(2\mu+1)}(x)\,\mathrm{d}x = \frac{1}{(2\mu+1)!}\sum_{k=m}^{n-1}\int_0^1 B_{2\mu+1}(x)\, f^{(2\mu+1)}(k+x)\,\mathrm{d}x\,. \tag{16}$$

$B_k(x)$ denotes the kth Bernoulli polynomial, and $B_k = B_k(0)$ the kth Bernoulli number. In our case we have

$$f(x) = \frac{1}{2x-1} \qquad \text{and} \qquad a_k = f(k) = \frac{1}{2k-1}\,.$$

Now we will derive an estimate of $R_\mu(m,n)$. For the kth derivative of f, one finds

$$f^{(k)}(x) = \frac{(-1)^k\, 2^k\, k!}{(2x-1)^{k+1}}\,.$$

Since all the functions $|f^{(k)}(x)|$, $k = 0, 1, 2, \ldots$, are strictly decreasing, for the terms in the sum of (16) we find

$$\left|\int_0^1 B_{2\mu+1}(x)\, f^{(2\mu+1)}(k+1+x)\,\mathrm{d}x\right| < \left|\int_0^1 B_{2\mu+1}(x)\, f^{(2\mu+1)}(k+x)\,\mathrm{d}x\right|.$$

The absolute value of each integral on the right-hand side of Eq. (16) is at most

$$\left|\int_0^1 B_{2\mu+1}(x)\, f^{(2\mu+1)}(m+x)\,\mathrm{d}x\right|$$

and there are $n-m$ of these integrals. Therefore

$$\big|R_\mu(m,n)\big| < \big|\widetilde{R}_\mu(m,n)\big| \tag{17}$$

where

$$\widetilde{R}_\mu(m,n) = \frac{n-m}{(2\mu+1)!}\int_0^1 B_{2\mu+1}(x)\, f^{(2\mu+1)}(m+x)\,\mathrm{d}x\,.$$

Eq. (17) is the desired estimate for $R_\mu(m,n)$. Furthermore, all

$$\int_0^1 B_{2\mu+1}(x)\, f^{(2\mu+1)}(k+x)\,\mathrm{d}x\,,\ k=m,\ldots,n-1,$$

have the same sign which is equal to the sign of $R_\mu(m,n)$, and to the sign of $\widetilde{R}_\mu(m,n)$. Using the integral

$$\int_m^n f(x)\,\mathrm{d}x = \int_m^n \frac{\mathrm{d}x}{2x-1} = \frac{1}{2}\left[\ln(2n-1)-\ln(2m-1)\right],$$

we get the explicit summation formula

$$\sum_{k=m}^{n} \frac{1}{2k-1} = \varphi_\mu(m,n) + R_\mu(m,n) \tag{18}$$

with the approximation

$$\begin{aligned}\varphi_\mu(m,n) = \frac{1}{2}\left(\ln(2n-1)-\ln(2m-1)+\frac{1}{2m-1}+\frac{1}{2n-1}\right)\\ -\sum_{j=1}^{\mu}\frac{2^{2j-1}B_{2j}}{2j}\left(\frac{1}{(2n-1)^{2j}}-\frac{1}{(2m-1)^{2j}}\right)\end{aligned} \tag{19}$$

and the remainder term

$$R_\mu(m,n) = -2^{2\mu+1}\sum_{k=m}^{n-1}\int_0^1 \frac{B_{2\mu+1}(x)}{[2(k+x)-1]^{2\mu+2}}\,\mathrm{d}x\,.$$

The explicit formula for the error bound is

$$\widetilde{R}_\mu(m,n) = -2^{2\mu+1}\,(n-m)\int_0^1 \frac{B_{2\mu+1}(x)}{[2(m+x)-1]^{2\mu+2}}\,\mathrm{d}x\,. \tag{20}$$

Using Eq. (18), we have

$$s(n) = s(m-1) + \varphi_\mu(m,n) + R_\mu(m,n)\,.$$

Hence an approximation for $s(n)$ is

$$\tilde{s}_{m,\,\mu} = s(m-1) + \varphi_\mu(m,n)\,.$$

As an example we estimate $s(68100151)$ for $\lambda = 10$, which is used in Eq. (10) to compute the value of $t(\lambda)$. We have

$$\begin{aligned}s(68100151) \approx \tilde{s}_{100001,\,3}(68100151) &= s(100000) + \varphi_3(100001, 68100151)\\ &= 10.000000007233082813117\ldots.\end{aligned}$$

λ	n
10	68100151
11	503195829
12	3718142208
13	27473561358
14	203003686106
15	1500005624924
16	11083625711271
17	81897532160125
18	605145459495141
19	4471453748222757
20	33039822589391676
21	244133102611731231
22	1803913190804074904
23	13329215764452299411
24	98490323038288832267
25	727750522131718025058

Table 2 Values of n

Eq. (10) requires that we raise the difference $s(n) - \lambda$ to the high power n. (Note the loss of precision that occurs when we perform this subtraction.) So, we may need to compute more accurate approximations $\varphi_\mu(m, n)$ using values of $\mu > 3$. Table 2 above and Table 3 below display n and the approximate values of $s(n)$ for $\lambda = 10, 11, \ldots, 25$. To obtain these values, we use $m = 100001$ and compute $s(m-1)$ to 100 decimal places, then use $\mu = 10$ to compute each n and $\varphi_\mu(m, n)$.

Note that if we compute the initial sum $s(m-1)$ to only D decimal places, then we can never compute $s(j)$ to more than D correct decimal places for any $j > m - 1$, *even if* the error estimate $\left|\widetilde{R}_\mu\right|$ is less than 10^{-D}.

For a given λ, we first compute n and the approximate value of $s(n)$. The next task is to compute the value of $I(\lambda)$ using Eq. (9). The value of $(s(n) - \lambda)^n$ can easily be obtained from the approximate value of $s(n)$ in Table 3, although for large n, we must use logarithms to prevent underflow.

6 Estimating the logarithm of the factorials

We can write

$$\ln t(\lambda) = n \ln(s(n) - \lambda) - (2n - 1) \ln 2 + \sigma(n)$$

with

$$\sigma(n) := \ln \frac{(2n)!}{n!^2} = \ln((2n)!) - 2\ln(n!) = \sum_{k=1}^{2n} \ln k - 2 \sum_{k=1}^{n} \ln k\,.$$

λ	$\tilde{s}_{m,\mu}(n)$	$\widetilde{R}_{\mu}(m,n)$
10	$10 + 7.2330828131174081549544093888189287562979322961080227530383865 9 \cdot 10^{-9}$	$2.10 \cdot 10^{-104}$
11	$11 + 1.9342969472157193824359222060920860745938666699351199611517044 7 \cdot 10^{-10}$	$1.56 \cdot 10^{-103}$
12	$12 + 2.8170475701700398606156216335922104758242021233575450606242827 3 \cdot 10^{-11}$	$1.15 \cdot 10^{-102}$
13	$13 + 1.5178452834334065797485545917289020886965907886920725717263202 4 \cdot 10^{-11}$	$8.50 \cdot 10^{-102}$
14	$14 + 1.2000435910160962912208044565237295521804126111721750249602618 3 \cdot 10^{-12}$	$6.28 \cdot 10^{-101}$
15	$15 + 2.1918027214988747090960676198940289109809303488543505647946301 0 \cdot 10^{-13}$	$4.64 \cdot 10^{-100}$
16	$16 + 4.0377670109254265093506208840414588864130287862673117338645250 0 \cdot 10^{-14}$	$3.43 \cdot 10^{-99}$
17	$17 + 3.9681161061091988001262156896829211999200705438985081243994589 5 \cdot 10^{-16}$	$2.54 \cdot 10^{-98}$
18	$18 + 6.4418462955235916712061651151307108967176903595484368355241024 0 \cdot 10^{-16}$	$1.87 \cdot 10^{-97}$
19	$19 + 4.4065883547028358507185382022388728562996883400850773168475519 1 \cdot 10^{-17}$	$1.38 \cdot 10^{-96}$
20	$20 + 2.9025868310489491349920307015316760066957706491685602092628720 4 \cdot 10^{-18}$	$1.02 \cdot 10^{-95}$
21	$21 + 7.3428066905705430683281842456395910206826195554807901623461364 6 \cdot 10^{-19}$	$7.56 \cdot 10^{-95}$
22	$22 + 1.3068356056770845920453738891212973149245847447188817166232937 9 \cdot 10^{-19}$	$5.58 \cdot 10^{-94}$
23	$23 + 3.4063384440895510901408320322419983991199965675874881595312543 9 \cdot 10^{-20}$	$4.13 \cdot 10^{-93}$
24	$24 + 3.7949965848604631831655558125977106217088878142367501476347262 3 \cdot 10^{-21}$	$3.05 \cdot 10^{-92}$
25	$25 + 1.1432548064658205122366981824665412932645819762422489580736204 9 \cdot 10^{-22}$	$2.25 \cdot 10^{-91}$

Table 3 Approximations for $s(n)$ with error bound for $m = 100001$ and $\mu = 10$

To get a good estimate for $\sigma(n)$, we will use the exact sum of $m-1$ initial terms. Therefore, we split the sums:

$$\sigma(n) = \left[\sum_{k=1}^{m-1} + \sum_{k=m}^{2n} - 2\left(\sum_{k=1}^{m-1} + \sum_{k=m}^{n}\right)\right] \ln k = -\left(\sum_{k=1}^{m-1} + \sum_{k=m}^{n} - \sum_{k=n+1}^{2n}\right) \ln k\,. \quad (21)$$

It remains to estimate $\sum_{k=m}^{n} \ln k$ and $\sum_{k=n+1}^{2n} \ln k$. Therefore, we apply the Euler–Maclaurin summation formula (15) with

$$f(x) = \ln x\,.$$

The derivatives are given by

$$f^{(k)}(x) = \frac{(-1)^{k-1}\,(k-1)!}{x^k}\,, \quad k = 1, 2, \ldots\,. \quad (22)$$

Furthermore, we have

$$\int_m^n f(x)\,\mathrm{d}x = \int_m^n \ln x\,\mathrm{d}x = x(\ln x - 1)\Big|_m^n = n(\ln n - 1) - m(\ln m - 1)\,.$$

This yields

$$\sum_{k=m}^{n} \ln k = \psi_\mu(m,n) + R_\mu^*(m,n)$$

with the approximation

$$\psi_\mu(m,n) = n(\ln n - 1) - m(\ln m - 1) + \frac{\ln m + \ln n}{2} \\ + \sum_{j=1}^{\mu} \frac{B_{2j}}{2j(2j-1)} \left(\frac{1}{n^{2j-1}} - \frac{1}{m^{2j-1}} \right) \tag{23}$$

and the remainder term

$$R^*_\mu(m,n) = \frac{1}{2\mu+1} \sum_{k=m}^{n-1} \int_0^1 \frac{B_{2\mu+1}(x)}{(k+x)^{2\mu+1}}\, \mathrm{d}x \,.$$

Since the absolute values of all derivatives in Equation (22) are strictly decreasing, we find the error estimate

$$\left| R^*_\mu(m,n) \right| < \left| \widetilde{R}^*_\mu(m,n) \right|$$

with

$$\widetilde{R}^*_\mu(m,n) = \frac{n-m}{2\mu+1} \int_0^1 \frac{B_{2\mu+1}(x)}{(m+x)^{2\mu+1}}\, \mathrm{d}x \,. \tag{24}$$

It follows that the approximation for (21) is given by

$$\tilde{\sigma}_{m,\,\mu}(n) = -\sum_{k=1}^{m-1} \ln k - \psi_\mu(m,n) + \psi_\mu(n+1, 2n) \tag{25}$$

with the error bound

$$\left| \tilde{\sigma}_{m,\,\mu}(n) - \sigma(n) \right| < \left| \widetilde{R}^*_\mu(m,n) \right| + \left| \widetilde{R}^*_\mu(n+1, 2n) \right| . \tag{26}$$

7 Results for $\lambda \geq 10$

We can now put all of this together to compute the values of $t(\lambda)$ for many λ values. For example, Table 4 shows the value $t(\lambda)$ for $\lambda = 10 \ldots 25$. All digits shown below are correct, rounded in the last decimal place.

The original draft of this paper [2] contains all of the *Mathematica* code that gives these results. Although only results for integer values of λ are shown here, this code computes $t(\lambda)$ for real $\lambda \geq 1$.

To obtain the values in Table 4, we used $m = 100001$ and $\mu = 10$ in Eq. (19). We summed the first $m - 1 = 100000$ terms with Eq. (2), and then used Eq. (23) with $\mu = 5$ and $m = 100001$ to compute logs of factorials. We computed the sum of the logs of the first $m - 1$ integers to 100 decimal places. The integral in Eq. (24) was computed in *Mathematica* with the option `WorkingPrecision = 40`.

On a 2010-vintage laptop, it took about 9 seconds for the code in [2] to generate Table 4. This includes about 3.7 seconds to compute the sums of 100000 terms in Eq. (2), and about 2.4 seconds to compute the sum of the logarithms of the integers ≤ 100000.

λ	$t(\lambda)$
10	$9.649273600428684463479553120939810530923242208735398 \cdot 10^{-554381308}$
11	$7.57929806494947536128349934756162195412431861759227 \cdot 10^{-4887781043}$
12	$5.30200436015724605246826614752108917188558325098544 \cdot 10^{-39227165565}$
13	$1.52739916984845667363367296109645541442392755493153 \cdot 10^{-297230209953}$
14	$6.617345077783595182168242992545965700461478406777 \cdot 10^{-2419966945909}$
15	$5.26019597269976433379615815051550875066124252042 \cdot 10^{-18988869014266}$
16	$4.06751521421327233190115950829638686972451899823 \cdot 10^{-148452517153987}$
17	$2.8703074957720537216132995767534053015103162770 \cdot 10^{-1261337931785960}$
18	$1.46932966274512803735093876340436661798499994 \cdot 10^{-9192758406970262}$
19	$7.36887339695623805028019042180415757921528157 \cdot 10^{-73134639260589997}$
20	$5.8024461422390775663611817270349845938468954 \cdot 10^{-579426465025122292}$
21	$1.3869021709986676325063938918439160007102035 \cdot 10^{-4427143349945912840}$
22	$4.10151245193385022941804060193305405447094 \cdot 10^{-34064698104956009918}$
23	$1.71715972357092319138607947022428968556534 \cdot 10^{-259489336406929338805}$
24	$1.4982758030623762036996263232893870122517 \cdot 10^{-2011250066953860707590}$
25	$2.723848647528233561685563127899349771647 \cdot 10^{-15968197862152240928105}$

Table 4 Values of $t(\lambda)$ for the evaluation of the integrals $I(\lambda)$

These calculations can also be extended beyond $\lambda = 25$. For example, running the code with the above parameters gives, for $\lambda = 40$,

$$t(40) \approx 1.8758610 \cdot 10^{-266134053348172015148849587491648267}.$$

As λ increases, more and more of the significant digits in the calculation are consumed in the exponent.

8 Conclusion

We show how to evaluate highly-oscillatory integrals involving the sinc function. Our procedure has two main ingredients: a result of David and Jon Borwein, and the Euler–Maclaurin summation formula. We show how to evaluate these integrals to high precision, and avoid overflow and underflow, even though intermediate results are well beyond the range of ordinary floating-point arithmetic.

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