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# Rotationally symmetric tilings with convex pentagons and hexagons

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## 1 Introduction

Recent news in 2015 [6] made a nearly hundred-years-old problem popular again: the complete characterization of all convex pentagons that tile the Euclidean plane monohedrally (i.e., filling the plane completely without gaps or overlaps by using only congruent copies of one special tile). There is no hint that this characterization can be fully done in the near future, so only a systematic catalog of the existing tilings can be given (see, e.g., [1] or [2]). But why are pentagons so interesting mathematically? The reason is simple: For all other convex polygons the question is settled. So, the pentagons remained as an open problem.

In most papers about this subject the emphasis is put on periodic tilings. One reason might be the fact that all known families of pentagons tiling the plane are able to generate periodic tilings but only few of them also deliver rotationally symmetric ones (e.g., [3] or at the end of [4]). It is good practice to denote the type of symmetry in the catalogs of tilings. But we also could view the problem from the other direction: For which types of symmetry is a

Parkettierungen der Ebene gehören zu den beliebtesten und reichhaltigsten Themen der ebenen Geometrie. In jüngster Zeit rückten Parkettierungen mittels kongruenter konvexer Fünfecke ins Zentrum des Interesses, da nach langer Zeit eine weitere Zerlegung dieser Art gefunden worden war. Alle 15 bisher veröffentlichten Zerlegungen sind periodisch, d. h. sie wiederholen sich wie ein Tapetenmuster nach einem festen Rhythmus. Im vorliegenden Artikel wird mit der Rotation eine andere Art von Symmetrie untersucht, aber ebenfalls mit kongruenten konvexen Fünfecken. Es wird gezeigt, dass alle denkbaren Rotationssymmetrien durch Zerlegungen mit Fünfecken einer bestimmten Klasse repräsentiert werden können. Darüber hinaus ergeben sich visuell attraktive Spiralstrukturen, die in all diesen Zerlegungen auftreten.

Figure 1 Examples for $\mathbf{C}_3$ (taken from [5]) and $\mathbf{D}_4$ symmetry

monohedral tiling with convex pentagons possible? We will answer this question positively for all types of rotational symmetry.

In our case the Schoenflies notation for symmetry is the easiest one, which uses $\mathbf{C}_n$ for $n$-fold rotational symmetry without reflection and $\mathbf{D}_n$ for additional $n$ axes of reflection (e.g., see Figure 1).

Regarding the periodic case, only four types of rotational symmetry are possible: $n = 2$, 3, 4 or 6. It is interesting to observe that all of these cases occur in the catalog of known periodic pentagon tilings. For the non-periodic case, which is studied here, we will see that we can go further. From related work (e.g., [11], [12]) tilings are known with arbitrary rotational symmetry but they are generated by rhombs or triangles as prototiles. Here we define a class of pentagons, from which one can generate the proposed symmetry types. The following properties define the class:

**Property 1.** *Let $A, B, C, D, E$ be the inner angles in anti-clockwise order of a pentagon $P$ and $a, b, c, d, e$ the edges with $a$ ending at corner $A$ and so forth. $P$ is regarded to have property 1 if all inner angles $< 180°$ and $|b| = |c| = |a|+|d|$ and $D+E = 180°$ and $C \mid 360° \vee B \mid 360°$ where $|a|$ denotes the length of side $a$.* (See Figure 2 for an example.)

Obviously, this pentagon class is a subclass of "type 1" from [2] and has three degrees of freedom: The angles $B$, $C$ and $D$ can be chosen within certain ranges and constraints but independently from each other. The other angles and proportions are dependent from these three.

## 2 Results

Without loss of generality we assume for simplicity that angle $B$ divides 360°. (If $B$ and $C$ are interchanged, the following theorem also holds with equivalent proof.)

**Theorem.** *For a given natural number $n > 2$ any property-1-pentagon with $B = 360°/n$ tiles the Euclidean plane with $n$-fold rotational symmetry. If $C = 180° - B/2$ and $D = 90°$, the plane can be tiled with type $\mathbf{D}_n$, otherwise with $\mathbf{C}_n$. For $n = 2$ or $n = 1$ such a tiling is possible with any $B < 180°$, in particular with property-1-pentagons.*

*Proof.* We start with the case $n > 2$. First, we should note that each pentagon $P$ with property 1 has parallel sides $d$ and $a$. So, we can take a copy of $P$ and glue both together at side $e$ to form a hexagon, in most cases not a regular one, but equilateral and with parallel sides.

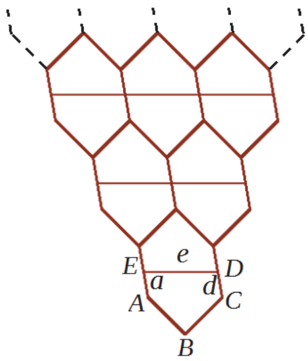


Figure 2 A sector formed by congruent hexagons

Putting copies of these hexagons together in form of growing rows, one can fill a kind of sector of the plane. The outer shape of this sector is fully characterized by the angles, since the length of the line segments is always $|b|$. Viewing this sector in upright position as in Figure 2, we see that the left border is a zig-zag-line with alternating inner angles $A$ and $180° - A$, the right border has angles $C$ and $180° - C$ alternating.

Now we take a copy of this sector and connect it to the first one as shown in Figure 3. Between both sectors a gap remains which has the following characterization: The innermost point has outer angle $A + C$, which is $540° - B - D - E = 360° - B$, so the inner angle of the gap is $B$. The left border is an equilateral zig-zag with alternating angles $C$ and $180° - C$, the right border has angles $A$ and $180° - A$ alternating.

It follows that the original sector from Figure 2 perfectly fits into the gap after a reflection operation at the vertical axis and a suitable rotation. Such a reflection plus rotation is allowed, since all parts within a sector remain congruent under this operation.

With these operations we have to generate $2n$ sectors to fill the $360°$ angle around the origin of the plane and the $n$ gaps. By construction, the symmetry $\mathbf{C}_n$ is obvious. To introduce

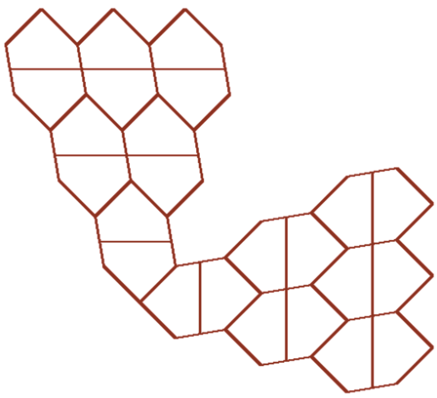

Figure 3 Two sectors with a gap

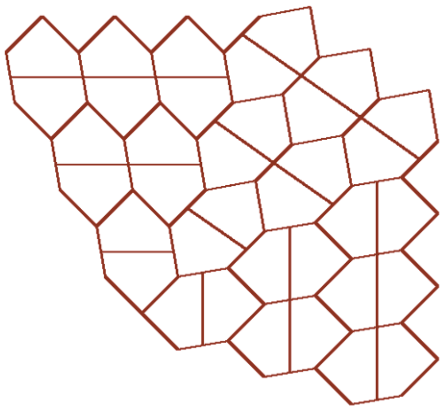

Figure 4 Another sector copy fills the gap after reflection plus rotation

the additional reflection axes needed for $\mathbf{D}_n$ , we just have to view the case $C = 180° - B/2$ and $D = 90°$ (see Fig. 5). Here the sector itself has reflection symmetry. Finally, we have $2n$ sector copies but each two of them share the same axis. So, there are $n$ axes for $\mathbf{D}_n$. (For a full view, see also the examples in Section 3.)

How can we see that our tilings are really plane filling? To prove this, we can regard the tiling of each sector as a sequence of rows with a growing number of hexagons. Consider those rows with a fixed number of $m$ ($> 1$) hexagons, which – as a union – always forms a closed ring built out of $2nm$ hexagons. Then each of the described tilings can be seen as a growing series of rings around the centre with each ring consisting of hexagons. The first union with $m = 1$ is topologically equivalent to a disk. Each further ring enlarges this disk, the size of which must grow with linearly increasing diameter, since the shape of the hexagons cannot degenerate. Each new ring fits to the previous one without gap. So, any point on the plane will be covered by this growing disk after enough rings were added consecutively.

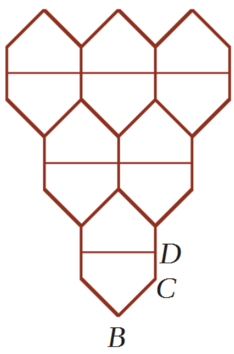


Figure 5 Sector with reflection symmetry

But is it always possible to construct a pentagon with inner angles $< 180°$? It is, since $120° \geq B > 0$ for all $n > 2$. Then $A$ and $C$ can be chosen, e.g., $(360° - B)/2 \pm B/4$ respectively, which is smaller than 180°. The choice of $D$ can be $270° - B/2 - C$, which is also between 0 and 180°.

Finally we have to discuss the cases $n = 2$ and $n = 1$. Here the above construction will result in rectangles instead of hexagons, which will not deliver proper pentagons with angles $< 180°$. So, we should look into the existing catalogue for pentagon tilings. The most simple ones, called "houses tiling", will help. Simple examples for $n = 2$ are shown. The point of symmetry is marked by a dot.

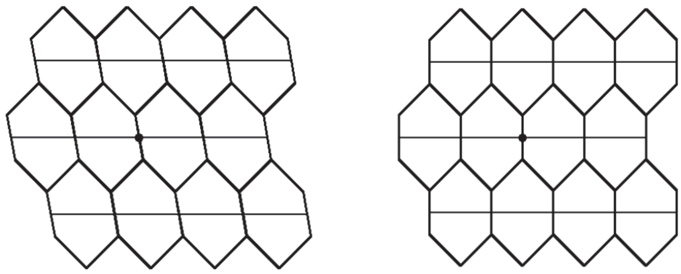

Figure 6 Examples for $\mathbf{C}_2$ and $\mathbf{D}_2$ symmetry

The case $\mathbf{C}_1$ means "neither rotational nor reflexion symmetry" (see Figure 7 left) and $\mathbf{D}_1$ has only one symmetry axis and no rotational symmetry (see the right part of Figure 7).

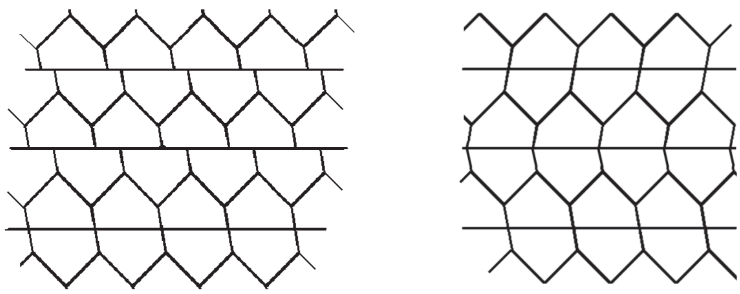

Figure 7 Examples for $\mathbf{C}_1$ and $\mathbf{D}_1$ symmetry

So, all symmetry types $\mathbf{C}_n$ and $\mathbf{D}_n$ were constructed with pentagons having property 1. □

## 3 Examples

We should not finish the paper without showing some of the nice tilings resulting from the above construction. Figure 8 represents $\mathbf{C}_5$, Figure 9 $\mathbf{D}_7$ symmetry.

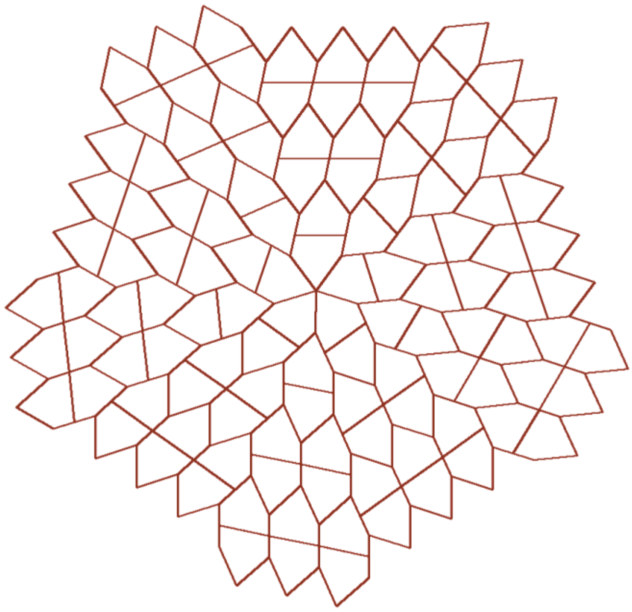

Figure 8 $\mathbf{C}_5$ symmetry with $(A,\ B,\ C,\ D,\ E) = (132°, 72°, 156°, 78°, 102°)$

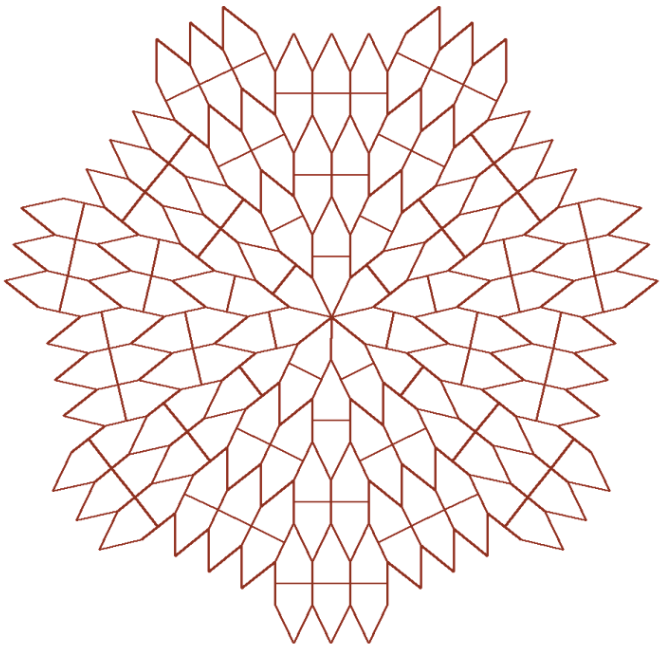

Figure 9 Example for $\mathbf{D}_7$ symmetry

## 4 Spirals

Another visually attractive property of the above construction is the following. Any tiling according to Section 2 with $n$-fold rotational symmetry and $n > 1$ can also be regarded as spiral tiling with $n$ congruent arms (see [10] for a precise definition of the term "spiral tiling"). To give an example, we can identify one of the seven spiral arms within the $\mathbf{D}_7$ tiling from the above Figure 9.

Take one of the innermost hexagons for the spiral's begin and walk outward obeying the rule: Find the dividing line in the hexagon's middle and choose the neighboring hexagon at the line's right endpoint as next hexagon for the spiral (where "right" means "right viewed from the origin").

Figure 10 displays this partition into spiral arms. In the well-known book of Grünbaum and Shephard [7] it was put as an open question if spiral tilings exist with $n$ arms for any odd $n > 5$. Later several spiral tilings have been published (e.g., [8], [9] or [3]) but – to the author's knowledge – those with higher number of arms had non-convex tiles. Here we can show spirals with arbitrary number of congruent arms and with convex pentagons as prototiles.

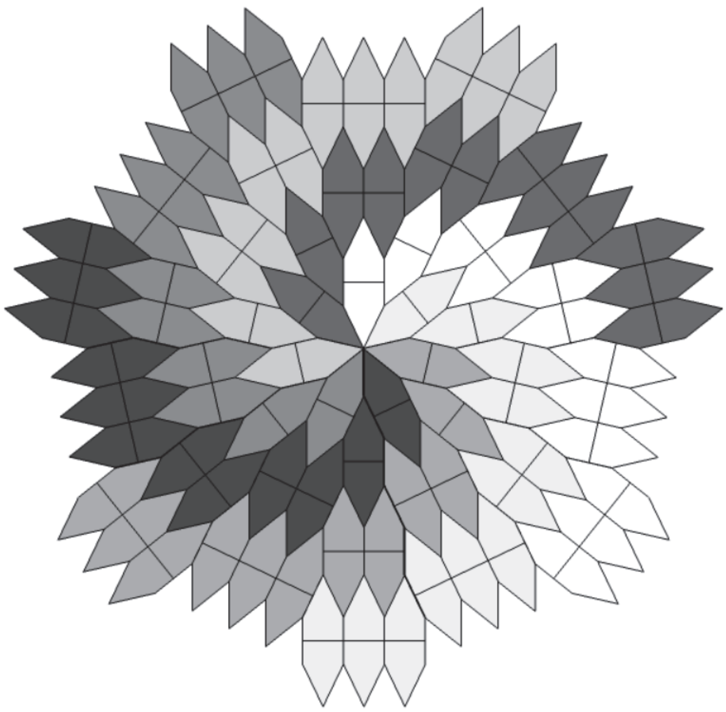

Figure 10 Demonstration of the tiling's spiral character

To summarize the results of this paper, we can state that for any rotational symmetry type $\mathbf{C}_n$ or $\mathbf{D}_n$ there is a tiling of the 2D plane with convex pentagons representing this symmetry. Each of these tilings with $n > 2$ can also be regarded as spiral tiling with $n$ congruent arms. By construction it is obvious that this holds for hexagons, as well.

After handling all rotational symmetric cases, there remains an open question in the periodic case. The author's presumption is that it will not be possible to represent all 2D symmetry types by convex pentagon tilings, unless degenerated pentagons are used (i.e., triangles or quadrangles).

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