Elem. Math. 72 (2017) 2 – 8
0013-6018/17/010002-7
DOI 10.4171/EM/317

Elemente der Mathematik

An alternate proof of Gerretsen's inequalities

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1 The inequality

The second degree inequality which establishes a fundamental relation between the semi-perimeter s on the one side, and the circumradius R and the inradius r of a triangle ABC on the other side is

$$16Rr - 5r^2 \leqslant s^2 \leqslant 4R^2 + 4Rr + 3r^2. \tag{1}$$

This double inequality, known as Gerretsen's inequality [6], is invaluable in the theory of triangle inequalities. The standard way of proving it is to calculate the squares of the distances from the incenter to the centroid and the orthocenter. It resembles the derivation of the Euler inequality $R \geqslant 2r$ from the Euler formula $OI^2 = R(R-2r)$ for the distance from the incenter I to the circumcenter O, [4]. Let G and H denote the centroid and the orthocenter of a triangle. Then

$$9GI^2 = s^2 - 16Rr + 5r^2,$$

and

$$HI^2 = 4R^2 + 4Rr + 3r^2 - s^2.$$

In der Dreiecksgeometrie gehören die Ungleichungen von Gerretsen zu den wichtigsten quadratischen Ungleichungen: Sie beschränken den halben Umfang bei gegebenem Um- und Inkreisradius von oben und von unten. Der übliche Beweis beruht auf dem Ausrechnen von Abständen zwischen ausgezeichneten Punkten des Dreiecks. In der vorliegenden Arbeit liefert der Autor einen weiteren, elementaren Beweis, indem er neben der bekannten Schurschen Ungleichung eine einfache, allgemeingültige Ungleichung für drei reelle Zahlen ins Spiel bringt. Darüber hinaus zeigt er, wie man einige bekannte Ungleichungen aus den Ungleichungen von Gerretsen folgern kann.

Since squares must be non-negative, the inequalities immediately follow. For the derivation of HI^2 see [8, p. 200]. Once HI^2 is determined, one can consider the triangle OHI and its Cevian GI. Using Euler's formula $OI^2 = R(R-2r)$, $OH^2 = 9R^2 - (a^2+b^2+c^2)$, the ratio $OG : GH = 1 : 2$ on the Euler line, and invoking Stewart's theorem, GI^2 is easily computed. Another way of proving the Gerretsen inequalities is by deducing them from the so-called fundamental inequality, [1],

$$\begin{aligned} &2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr} \\ &\quad \leqslant s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr}, \end{aligned}$$

whose proof is rather artificial and involved. Indeed,

$$\begin{aligned} &2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr} \\ &\quad = 16Rr - 5r^2 + \left(R - 2r - \sqrt{R^2-2Rr}\right)^2 \geqslant 16Rr - 5r^2, \end{aligned}$$

and

$$\begin{aligned} &2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr} \\ &\quad = 4R^2 + 4Rr + 3r^2 - \left(R - 2r - \sqrt{R^2-2Rr}\right)^2 \leqslant 4R^2 + 4Rr + 3r^2. \end{aligned}$$

We give a proof of the LHS inequality of (1) based on the well-known Schur inequality. For the RHS inequality of (1) we use a simple inequality for three real numbers and the same trigonometric identity used in the standard proof.

2 Lemmas

Lemma 1 (Schur's inequality)**.** *For three positive numbers x, y and z and all $a \geqslant 0$ it holds*

$$x^a(x-y)(x-z) + y^a(y-x)(y-z) + z^a(z-x)(z-y) \geqslant 0,$$

with equality if and only if $x = y = z$.

For an easy proof see [11, p. 83].

Let

$$T_1 := x + y + z, \quad T_2 := xy + yz + zx, \quad T_3 := xyz.$$

For $a = 1$ the Schur inequality can be rewritten as

$$T_1^3 - 4T_1T_2 + 9T_3 \geqslant 0.$$

Lemma 2. *For three real numbers a, b and c it holds*

$$(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2) \leqslant (-a+b+c)^2(a-b+c)^2(a+b-c)^2.$$

Proof. Let $a \leqslant b \leqslant c$ and assume that $a^2+b^2-c^2 \geqslant 0$, since the other cases are trivial. Then by multiplying the three inequalities

$$\begin{aligned}(-a^2+b^2+c^2)(a^2-b^2+c^2) &\leqslant (-a+b+c)^2(a-b+c)^2,\\ (a^2-b^2+c^2)(a^2+b^2-c^2) &\leqslant (a-b+c)^2(a+b-c)^2,\\ (-a^2+b^2+c^2)(a^2+b^2-c^2) &\leqslant (-a+b+c)^2(a+b-c)^2,\end{aligned}$$

and taking the square root, we get the desired inequality. For the first inequality, we have

$$\begin{aligned}(-a+b+c)^2(a-b+c)^2 &- (-a^2+b^2+c^2)(a^2-b^2+c^2)\\ &= \left(c^2-(a-b)^2\right)^2 - \left(c^4-(a^2-b^2)^2\right)\\ &= (a-b)^2\left[(a-b)^2-2c^2+(a+b)^2\right]\\ &= 2(a-b)^2(a^2+b^2-c^2) \geqslant 0,\end{aligned}$$

and similarly for the others. □

Lemma 3. *For the product of the cosines of the angles in a triangle it holds*

$$\cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}.$$

Proof. One proof is given in the excellent book [9, p. 56] where it is shown that the cosines are roots of the polynomial

$$4R^2x^3 - 4R(R+r)x^2 + (s^2+r^2-4R^2)x + (2R+r)^2 - s^2 = 0.$$

By Vieta's formula follows the claim. A more direct proof follows from the trigonometric identity

$$\cos A\cos B\cos C = \frac{1}{2}(\sin^2 A+\sin^2 B+\sin^2 C) - 1, \tag{2}$$

the Law of Sines, $\sin A = a/(2R)$, and the algebraic identity for the sum of the squares of the sides

$$a^2+b^2+c^2 = 2(s^2-4Rr-r^2). \tag{3}$$

The trigonometric identity (2) is equivalent to

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C = 1.$$

The last one is true, since

$$\begin{aligned}\cos^2 A+\cos^2 B+\cos^2 C &- 1\\ &= \frac{1+\cos 2A}{2}+\frac{1+\cos 2B}{2}+\cos^2(A+B)-1\\ &= \cos(A+B)\left[\cos(A-B)+\cos(A+B)\right]\\ &= -2\cos A\cos B\cos C.\end{aligned}$$

For the algebraic identity see [9, p. 52]. □

3 The proof

Every inequality for three positive $x, y, z > 0$, with the substitution

$$a = y + z, \quad b = z + x, \quad c = x + y,$$

can be translated to an inequality for the sides of a triangle a, b, c, and vice versa. Then by invoking Heron's formula for the area $A = \sqrt{s(s-a)(s-b)(s-c)}$, $A = rs = abc/(4R)$ and the identity $(x+y)(y+z)(z+x) = (x+y+z)(xy+xz+zx) - xyz$, we get for the elements of the triangle in terms of T_1, T_2, T_3

$$s = T_1, \quad r^2 = \frac{T_3}{T_1}, \quad Rr = \frac{T_1 T_2 - T_3}{4T_1}.$$

Thus

$$\begin{aligned} s^2 - 16Rr + 5r^2 &= T_1^2 - 16\frac{T_1 T_2 - T_3}{4T_1} + 5\frac{T_3}{T_1} \\ &= \frac{T_1^3 - 4T_1 T_2 + 9T_3}{T_1} \geqslant 0, \end{aligned}$$

by the Schur inequality. That is the LHS of Gerretsen's inequality (1).

For the proof of the RHS we take a, b and c in Lemma 2 to be the sides of a triangle. By the Law of Cosines, $-a^2+b^2+c^2 = 2bc\cos A$ and similarly for the other multiples. Then Lemma 2 gives

$$\begin{aligned} 8a^2b^2c^2 \cos A \cos B \cos C &\leqslant 64(s-a)^2(s-b)^2(s-c)^2 \\ &= 64A^4/s^2 = 4a^2b^2c^2\frac{r^2}{R^2}. \end{aligned}$$

Now we apply Lemma 3 to the expression for cosines and obtain

$$s^2 - (2R + r)^2 \leqslant 2r^2,$$

which is the RHS of Gerretsen's inequality (1).

4 Equivalent forms

In this section we will give a few interesting equivalent forms of Gerretsen's inequality. It is remarkable that though not explicitly, the inequality has appeared almost a century before Gerretsen's publication. In 1870 M. Colins [3] gave the following inequality for the sides a, b, c of a triangle

$$2(a+b+c)(a^2+b^2+c^2) \geqslant 3(a^3+b^3+c^3+3abc).$$

It is equivalent to the LHS of (1) by (3) and the identity, see [9, p. 52]

$$a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2).$$

By (3) it follows that

$$24Rr - 12r^2 \leqslant a^2 + b^2 + c^2 \leqslant 8R^2 + 4r^2,$$

is an equivalent inequality to (1), and so is

$$4r(5R - r) \leqslant ab + bc + ca \leqslant 4(R + r)^2,$$

since

$$ab + bc + ca = \frac{1}{2}\left[(a + b + c)^2 - (a^2 + b^2 + c^2)\right] = s^2 + 4Rr + r^2. \tag{4}$$

The RHS of the last inequality can be rewritten in trigonometric form using the Law of Sines and the well-known identity

$$\cos A + \cos B + \cos C = \frac{R + r}{R}.$$

Then it becomes

$$\sin A \sin B + \sin B \sin C + \sin C \sin A \leqslant (\cos A + \cos B + \cos C)^2.$$

Another equivalent trigonometric form of the inequality is

$$\cos A \cos B \cos C \leqslant (1 - \cos A)(1 - \cos B)(1 - \cos C),$$

which comes as a byproduct from the derivation of HI^2 in [8].

5 Ono's, Blundon's and Hadwiger–Finsler inequality

In 1914 T. Ono conjectured [10] that for all triangles

$$27(-a^2 + b^2 + c^2)^2(a^2 - b^2 + c^2)^2(a^2 + b^2 - c^2)^2 \leqslant (4A)^6.$$

The conjecture was subsequently shown to be false in general, with the simple counterexample $a = 2$, $b = 3$, $c = 4$ and $A = 3\sqrt{15}/4$. However, it is true for acute triangles. The inequality from Lemma 2 can be rewritten as

$$(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)(a^2 + b^2 - c^2) \leqslant \frac{r}{s}(4A)^3.$$

Combining this with the well-known inequality $s \geqslant 3\sqrt{3}r$, we get

$$3\sqrt{3}(-a^2 + b^2 + c^2)(a^2 - b^2 + c^2)(a^2 + b^2 - c^2) \leqslant (4A)^3.$$

But for an acute triangle all the terms in the last inequality are positive and it can be squared, giving the Ono inequality.

Blundon's inequality is the following linear inequality, [1]

$$s \leqslant 2R + (3\sqrt{3} - 4)r.$$

It is a consequence of Gerretsen's inequality, since by the Euler inequality

$$\begin{aligned} s^2 &\leqslant 4R^2 + 4Rr + 3r^2 \\ &= \Big(2R + (3\sqrt{3} - 4)r\Big)^2 - r(12\sqrt{3} - 20)(R - 2r) \\ &\leqslant \Big(2R + (3\sqrt{3} - 4)r\Big)^2. \end{aligned}$$

We remark that an inequality of type $s \leqslant \lambda R + \mu r$ holds for all triangles only if it has the form

$$s \leqslant 2R + (3\sqrt{3} - 4)r + \alpha(R - 2r) + \beta r,$$

for some $\alpha, \beta \geqslant 0$. In this sense Blundon's inequality is the best possible linear inequality. Similarly, $s \geqslant 3\sqrt{3}r$ is the best possible linear inequality of type $s \geqslant \lambda R + \mu r$, see [2].

To conclude this note, we show that the celebrated Hadwiger–Finsler inequality [5], [7]

$$4\sqrt{3}A + Q \leqslant a^2 + b^2 + c^2 \leqslant 4\sqrt{3}A + 3Q, \tag{5}$$

with $Q := (a-b)^2 + (b-c)^2 + (c-a)^2$, can also be deduced from Gerretsen's inequalities. It holds $\sqrt{3}s \leqslant 4R + r$, since by the RHS of (1) and Euler's inequality

$$\begin{aligned} 3s^2 &\leqslant 3(4R^2 + 4Rr + 3r^2) \\ &= (4R + r)^2 - (4R + 4r)(R - 2r) \\ &\leqslant (4R + r)^2. \end{aligned}$$

Hence by (4)

$$\begin{aligned} a^2 + b^2 + c^2 - \Big[(a - b)^2 + (b - c)^2 + (c - a)^2\Big] \\ = 4(ab + bc + ca) - (a + b + c)^2 \\ = 4r(4R + r) \geqslant 4\sqrt{3}A, \end{aligned}$$

proving the LHS of (5). Similarly by the LHS of (1) and $s \geqslant 3\sqrt{3}r$

$$\begin{aligned} a^2 + b^2 + c^2 - 3\Big[(a - b)^2 + (b - c)^2 + (c - a)^2\Big] \\ = 16(ab + bc + ca) - 5(a + b + c)^2 \\ = 4(4r^2 + 16Rr - s^2) \\ \leqslant 36r^2 \leqslant 4\sqrt{3}A, \end{aligned}$$

which is the RHS of (5).

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