Elem. Math. 72 (2017) 15 – 23
0013-6018/17/010015-9
DOI 10.4171/EM/319



Elemente der Mathematik

# Alternative investigation into "Surprising sinc sums and integrals"

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## 1 Motivation

Time or frequency-dependent sincs (henceforth referred to as signals) are often used by students and researchers to analyze idealized models and obtain bounds for practical systems. The simplicity of the sinc *dual*, the rectangle signal, indeed makes it an attractive choice in order to deal with diverse problems such as interpolation, signal reconstruction, pulse shaping, intersymbol interference, and channel characterization to name a few. As a consequence, results related to the sinc become appealing to academicians in general and to the Signal Processing and Communication Engineering community in particular. In this work, we use the standard definition for the sinc, namely

$$\operatorname{sinc}(x) = \begin{cases} \sin(x)/x & \text{for } x \neq 0 \\ 1 & \text{for } x = 0. \end{cases}$$

An dieser Stelle wurde schon mehrfach über das verblüffende Phänomen der Baillie–Borwein-Integrale berichtet. Der Autor der vorliegenden Arbeit verwendet Parsevals Identität und die Faltung von Fourier-Reihen, um nochmals einen neuen Einblick zu gewinnen. Beim iterierten Falten eines Rechteckimpulses mit sich selber tritt ein horizontaler Verschmierungseffekt auf, der das Verhalten der Baillie–Borwein-Integrale anschaulich erklärt. Wir verweisen auch auf den Artikel von Hanspeter Schmid aus dem Jahr 2014, der in ähnlicher Weise mit iterierten Faltungen die vertikale Erosion des initialen Plateaus untersucht hat.

In 2006, this author independently presented some novel results related to sinc sums [1]. Shortly after, Baillie and the Borweins published results on surprising sinc sums and integrals [2] that drew his attention to their earlier reports in 2001 [3]. Recently, Schmid [4] presented a graphical proof of this remarkable sinc behavior introduced by the Borweins in 2001, namely that

$$\int_0^\infty \prod_{k=0}^{N} \operatorname{sinc}\left(\frac{t}{2k+1}\right) dt = \frac{\pi}{2} \tag{1}$$

for $N = 0, 1, 2, 3, 4, 5, 6$. However, for $N > 6$, the result of Eq. (1) suddenly becomes less than $\pi/2$. Schmid's proof is based on expressing the integral in Eq. (1) as the Fourier transform at the origin of the integrand being the convolution of $(N + 1)$ sincs. Hence, he repeatedly convolved two unit-area rectangular pulses but with different widths. The resulting initial plateau gets gradually eroded with more convolutions. He derived the same condition as the Borweins that makes the convolution peak (at the origin) drop below unity, hence causing the integral to break. Shortly after, Almkvist and Gustavsson [5] used the Poisson summation formula to establish the result of Eq. (1) amongst others. Baillie and the Borweins [2] also reported other interesting results related to sums and integrals involving sincs. Defining

$$S_N = \sum_{n=1}^{\infty} \operatorname{sinc}(n)^N \tag{2}$$

and

$$I_N = \int_0^\infty \operatorname{sinc}^N(f)\, df\ , \tag{3}$$

they reported that while $I_N$ is a rational multiple of $\pi$ for all non-zero integers $N$, $S_N$ is $-1/2$ plus a rational multiple of $\pi$ for $N = 1, 2, 3, 4, 5, 6$. However, $S_N$ suddenly changes to a polynomial in $\pi$ of degree $N$ for $N > 6$. More specifically,

$$S_N = I_N - \frac{1}{2} \tag{4}$$

for $N = 1, 2, 3, 4, 5, 6$ but not for $N > 6$. To explain this surprising sinc sum behavior, they established a general condition relating sinc sums to integrals, in addition to simple arguments using trigonometric identities and properties of Bernoulli polynomials.

Our aim in this work is to provide yet another graphically-illustrated and intuitively-simplistic explanation of the results in Eq. (4). We approach the problem with a signal processing background, motivated by Schmid's recent report [4], who too, had a similar approach to the problem. Even though our method bears some similarity to Schmid's, it is used in a different context that has been presented at an earlier time [1]. We use a repeated convolution of a rectangular pulse with itself in order to construct periodic waveforms. We use the Fourier series expansion along with Parseval's theorem [6] to provide alternative derivations of some of the sinc sums reported by Baillie and the Borweins. The sudden break in the sinc sum is clearly illustrated through an *aliasing* phenomenon that is due to the smearing effect of the repeated convolution. Hopefully, the reader shall find the dual use of the repeated convolution effect rather entertaining. While Schmid focused on the vertical effect of a repeated convolution, we shall focus on its horizontal effect in order to explain this interesting sinc behavior. Even though our illustrated approach is applied to only specific cases of the sinc sums, we hope that it is appealing enough with its clarity, simplicity and intuitive approach.

## 2 Alternative derivations

In the following we focus on the sinc sum $S_N$ of Eq. (2) for $N = 2, 4, 6$, and 8. We use the Fourier series expansion along with Parseval's theorem to provide alternative ways to evaluate these sums. In this process, we shall graphically illustrate through an aliasing effect the reason for which $S_8$ breaks the trend and changes from a rational multiple of $\pi$ to a polynomial in $\pi$ of degree 8. Our approach follows a previous work [1] wherein a series of four periodic waveform $s_i(t)$ $(i = 1, \ldots, 4)$ is constructed by duplicating base pulses $p_i(t)$ $(i = 1, \ldots, 4)$ at multiples of the period $T$ (taken to be unity). Hence,

$$s_i(t) = \sum_{n=-\infty}^{\infty} p_i(t-n) \quad \text{for } i = 1, \ldots, 4 \ \text{ and } \ t \in \mathbb{R}\,,$$

with $p_1(t)$ a rectangular pulse of unit amplitude and width $\alpha$ $(0 < \alpha \leq 1)$ defined by

$$p_1(t) = \begin{cases} 1 & \text{for } |t| \leq \alpha/2 \\ 0 & \text{for } |t| > \alpha/2 \end{cases} \tag{5}$$

and $p_2(t)$, $p_3(t)$, $p_4(t)$ are pulses resulting from the successive convolution of $p_1(t)$ with itself and are found to be

$$p_2(t) = \begin{cases} -|t| + \alpha & \text{for } |t| \leq \alpha \\ 0 & \text{for } |t| > \alpha \end{cases} \tag{6}$$

$$p_3(t) = \begin{cases} -t^2 + 3\alpha^2/4 & \text{for } |t| \leq \alpha/2 \\ (1/2)\left[|t| - 3\alpha/2\right]^2 & \text{for } \alpha/2 < |t| \leq 3\alpha/2 \\ 0 & \text{for } |t| > 3\alpha/2 \end{cases} \tag{7}$$

and

$$p_4(t) = \begin{cases} |t|^3/2 - \alpha t^2 + 2\alpha^3/3 & \text{for } |t| \leq \alpha \\ -|t|^3/6 + \alpha t^2 - 2\alpha^2|t| + 4\alpha^3/3 & \text{for } \alpha < |t| \leq 2\alpha \\ 0 & \text{for } |t| > 2\alpha. \end{cases} \tag{8}$$

Figure 1 depicts $s_1(t)$ with the base pulse $p_1(t)$ shown in bold. Signals $s_i(t)$ $(i = 2, 3, 4)$ have plots analogous to Figure 1 except that they use base pulses $p_i(t)$ $(i = 2, 3, 4)$. For the sake of conciseness, we omit the plots of $s_i(t)$ $(i = 2, 3, 4)$ and instead we depict in Figure 2 the normalized base pulses $p_2(t)$, $p_3(t)$, and $p_4(t)$ each respectively divided by $\alpha$, $\alpha^2$, and $\alpha^3$ (for $\alpha = 0.2$). Note that for all four signals $s_i(t)$ $(i = 1, \ldots, 4)$, $\alpha$ should not exceed a certain value in order for their repeating base pulses not to alias. Going

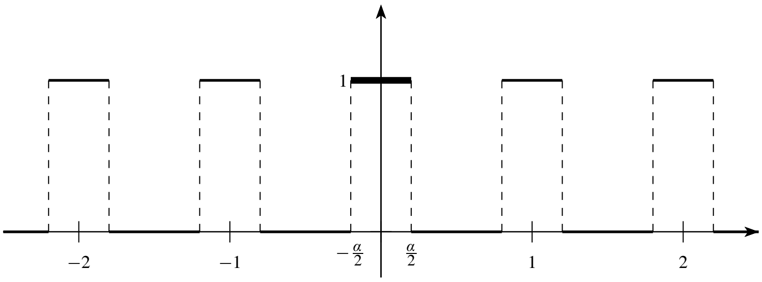


Fig. 1 Periodic signal $s_1(t)$ showing in bold the base pulse $p_1(t)$.

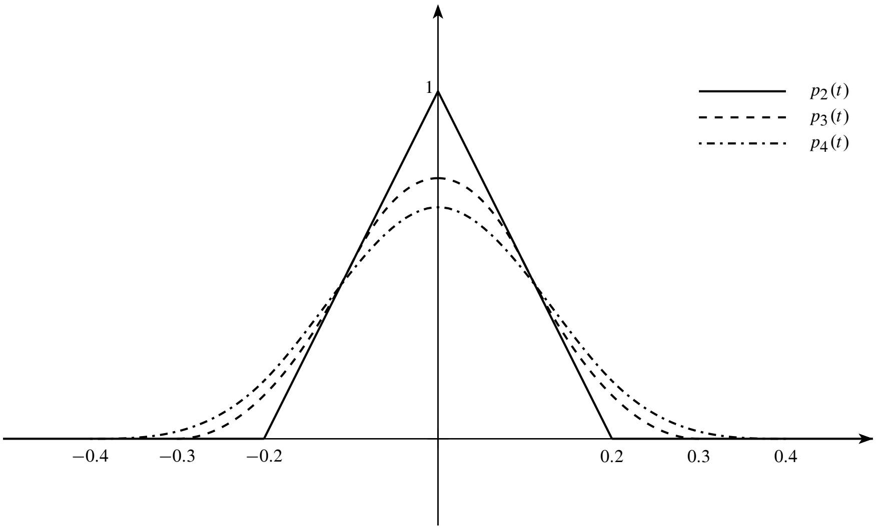


Fig. 2 Base pulses $p_2(t)$, $p_3(t)$, and $p_4(t)$ (respectively scaled by $1/\alpha$, $1/\alpha^2$, and $1/\alpha^3$) generated by the successive self-convolution of the rectangular base pulse $p_1(t)$ for $\alpha = 0.2$.

from $s_1(t)$ to $s_4(t)$, the range of $\alpha$ for alias-free signals is progressively reducing since each convolution widens the resulting pulse by $\alpha$. Hence, the condition on $\alpha$ for alias-free signal $s_i(t)$ ($i = 1, \ldots, 4$) is $0 < \alpha \leq 1/i$.

It is easy to check that the $n$th coefficient ($n$ is integer) of the Fourier series expansion of $s_1(t)$ is $c_{1n} = \alpha \operatorname{sinc}(\alpha\pi n)$. Using the convolution property of the Fourier series [6], the coefficients $c_{in}$ of signals $s_i(t)$ ($i = 2, 3, 4$) each time keep multiplying by $\alpha \operatorname{sinc}(\alpha\pi n)$. Hence, $c_{in}$ is directly found to be

$$c_{in} = \alpha^i \operatorname{sinc}^i(\alpha\pi n) \quad \text{for } i = 1, \ldots, 4. \tag{9}$$

Parseval's theorem states that

$$\int_{-1/2}^{1/2} |s_i(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_{in}|^2 \quad \text{for } i = 1, \ldots, 4 \tag{10}$$

where the modulus on both sides of Eq. (10) may be omitted since $c_{in}$ is real owing to the fact that $s_i(t)$ is real even symmetric [6]. In the absence of aliasing, $s_i(t) = p_i(t)$ for $|t| \leq 1/2$. Hence, using Eqs. (5)–(8), the LHS of Eq. (10) for $i = 1, \ldots, 4$ respectively evaluates to $\alpha$, $2\alpha^3/3$, $11\alpha^5/20$, and $151\alpha^7/315$. Using Eqs. (9) and (10), the even symmetry of $\mathrm{sinc}(\cdot)$, and the fact that $\mathrm{sinc}(0) = 1$, we get for $i = 1$

$$\sum_{n=-\infty}^{\infty} \alpha^2 \,\mathrm{sinc}^2(\alpha\pi n) = \alpha^2 \left[1 + 2\sum_{n=1}^{\infty} \mathrm{sinc}^2(\alpha\pi n)\right] = \alpha \quad \text{for } 0 < \alpha \leq 1$$
$$\Leftrightarrow \quad \hat{S}_2(\alpha) \equiv \sum_{n=1}^{\infty} \mathrm{sinc}^2(\alpha\pi n) = \frac{1}{2\alpha} - \frac{1}{2} \quad \text{for } 0 < \alpha \leq 1. \tag{11}$$

Similarly for $i = 2$, we get

$$\hat{S}_4(\alpha) \equiv \sum_{n=1}^{\infty} \mathrm{sinc}^4(\alpha\pi n) = \frac{1}{3\alpha} - \frac{1}{2} \quad \text{for } 0 < \alpha \leq 1/2 \tag{12}$$

and for $i = 3$

$$\hat{S}_6(\alpha) \equiv \sum_{n=1}^{\infty} \mathrm{sinc}^6(\alpha\pi n) = \frac{11}{40\alpha} - \frac{1}{2} \quad \text{for } 0 < \alpha \leq 1/3 \tag{13}$$

and finally for $i = 4$

$$\hat{S}_8(\alpha) \equiv \sum_{n=1}^{\infty} \mathrm{sinc}^8(\alpha\pi n) = \frac{151}{630\alpha} - \frac{1}{2} \quad \text{for } 0 < \alpha \leq 1/4. \tag{14}$$

To relate our derivations to those of Baillie et al. [2], we consider the special case of $\alpha = 1/\pi$. Hence, referring to Eqs. (2) and (11)–(14), $S_N$ is given by $\hat{S}_N(1/\pi)$ for $N = 2, 4, 6$, but not $N = 8$. This is because the non-aliasing conditions on $\alpha$ given at the end of Eqs. (11)–(14) are true for $N = 2, 4, 6,$ but fail for $N = 8$ since $1/4 < 1/\pi < 1/3$. Hence, we get

$$S_2 = \hat{S}_2(1/\pi) = \pi/2 - 1/2$$
$$S_4 = \hat{S}_4(1/\pi) = \pi/3 - 1/2$$
$$S_6 = \hat{S}_6(1/\pi) = 11\pi/40 - 1/2.$$

However,

$$S_8 = \hat{S}_8(1/\pi) = 151\pi/630 - 1/2 + \epsilon \tag{15}$$

where $\epsilon$ is an excess term resulting from aliasing. We now set to illustrate this sudden change in $S_8$ by extending $\hat{S}_8(\alpha)$ for $1/4 < \alpha \leq 1/3$ (that we shall denote by $\hat{\hat{S}}_8(\alpha)$) leading to the evaluation of $\epsilon$ as a polynomial in $\pi$ (as reported by Baillie and the Borweins [2]). For $1/4 < \alpha \leq 1/3$ the periodic signal $s_4(t)$ now undergoes an aliasing phenomenon. Let $s_{4A}(t)$ denote this aliased signal. Assume that $\alpha = \delta + 1/4$ , where $0 < \delta \leq 1/12$. Figure 3 illustrates $s_{4A}(t)$ for the case $\delta = 0.08$. For clarity, the vertical axis is scaled by $1/\alpha^3$. It is important to note that for $-1/2 \le t \le 1/2$, $s_{4A}(t)$ is not equal to $p_4(t)$ but rather to $p_{4A}(t)$ (highlighted in bold in Figure 3) given by

$$p_{4A}(t) = \begin{cases} |t|^3/2 - \alpha t^2 + 2\alpha^3/3 & \text{for } |t| \le \alpha \\ -|t|^3/6 + \alpha t^2 - 2\alpha^2|t| + 4\alpha^3/3 & \text{for } \alpha < |t| \le (1/2) - 2\delta \\ 2\delta t^2 - 2\delta|t| + 8\delta^3/3 + \delta/2 & \text{for } (1/2) - 2\delta < |t| \le 1/2 \\ 0 & \text{for } |t| > 1/2 \end{cases} \tag{16}$$

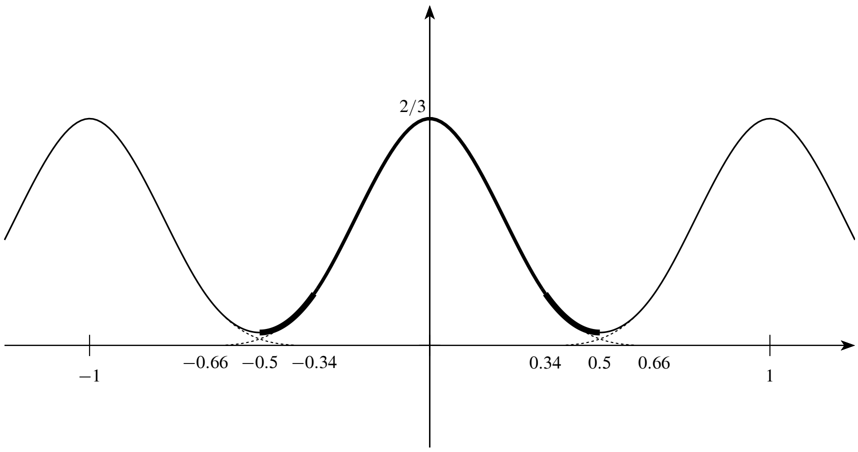


Fig. 3 Periodic signal $s_{4A}(t)$ illustrating the aliasing effect with $p_{4A}(t)$ highlighted in bold line. The extra-bold lines correspond to the aliased region that is behind the excess term $\epsilon$ causing the sudden break in $S_8$. In this plot, $\alpha = 0.33$ , $\delta = 0.08$, and the vertical axis is scaled by $1/\alpha^3$.

The last non-zero term in the definition of $p_{4A}(t)$ in Eq. (16) is highlighted by an extra-bold line in Figure 3. It is this aliasing region that is exactly responsible for the excess term $\epsilon$ in Eq. (15) which is behind this sudden break in $S_8$. It turns out that the aliasing does not affect the Fourier series coefficients of $s_{4A}(t)$ [6] (given by Eq. (9) as $c_{4n} = \alpha^4 \,\mathrm{sinc}^4\,(\alpha\pi n)$), but changes the LHS of Eq. (10) which is now given by

$$\begin{aligned} \int_{-1/2}^{1/2} s_{4A}^2(t)dt &= \int_{-1/2}^{1/2} p_{4A}^2(t)dt \\ &= \frac{151\alpha^7}{315} + \frac{1}{2520}\big[-1 + 28\alpha - 336\alpha^2 + 2240\alpha^3 - 8960\alpha^4 + 21504\alpha^5 \\ &\qquad - 28672\alpha^6 + 16384\alpha^7\big] \\ &= \frac{151\alpha^7}{315} + \frac{1}{2520}(-1+4\alpha)^7 . \end{aligned} \tag{17}$$

Using Eq. (17) and the fact that $c_{4n} = \alpha^4 \,\mathrm{sinc}^4\,(\alpha\pi n)$ in Eq. (10), we get

$$\hat{\hat{S}}_8(\alpha) = \sum_{n=1}^{\infty} \mathrm{sinc}^8(\alpha\pi n) = \frac{151}{630\alpha} - \frac{1}{2} + \frac{(-1+4\alpha)^7}{5040\alpha^8} \quad \text{for } \frac{1}{4} < \alpha \le \frac{1}{3} . \tag{18}$$

Comparing Eqs. (14) and (18), we note that the excess term resulting from aliasing turns out to be a polynomial of $1/\alpha$ and is given by

$$\hat{\epsilon}(\alpha) = \frac{(-1+4\alpha)^7}{5040\alpha^8} \quad \text{for } \frac{1}{4} < \alpha \leq \frac{1}{3}\,. \tag{19}$$

In particular, for the special case of $\alpha = 1/\pi$, the excess term $\epsilon$ in the evaluation of $S_8$ in Eq. (15) is given by (19) as

$$\epsilon = \hat{\epsilon}(1/\pi) = \frac{\pi(4-\pi)^7}{5040} \tag{20}$$

which is a polynomial in $\pi$ of order 8, as stated by Baillie and the Borweins. In reference to their derivations [2, Eqs. (26)–(35)], we may follow their procedure to elaborate on $S_8$ (instead of $S_7$) and find

$$\begin{aligned} S_8 &= \sum_{n=1}^{\infty} \frac{\operatorname{sinc}^8(n)}{n^8} \\ &= -128\,\pi^8 \left[ \frac{35}{128}\phi_8(0) - \frac{7}{16}\phi_8\left(\frac{2}{2\pi}\right) + \frac{7}{32}\phi_8\left(\frac{4}{2\pi}\right) \right. \\ &\quad \left. - \frac{1}{16}\phi_8\left(\frac{6}{2\pi}\right) + \frac{1}{128}\phi_8\left(\frac{8-2\pi}{2\pi}\right) \right] \end{aligned} \tag{21}$$

where $\phi_8(x)$ is the 8th normalized Bernoulli polynomial given by

$$\phi_8 = -\frac{1}{1209600} + \frac{x^2}{60480} - \frac{x^4}{17280} + \frac{x^6}{8640} - \frac{x^7}{10080} + \frac{x^8}{40320}. \tag{22}$$

It is easy to check that the substitution of Eq. (22) in Eq. (21) expresses $S_8$ as an 8th order polynomial in $\pi$ that is identical to $\hat{\hat{S}}_8\,(1/\pi)$ given by Eq. (18). In Eq. (21), "8" in the argument of the last term was replaced by $(8-2\pi)$ because the numerators of the arguments in the Bernoulli polynomial must not exceed $2\pi$. As pointed out by Baillie and the Borweins, it is exactly this fact that explains the sudden change in $S_8$ because $8 > 2\pi$. Hence, the excess term $\epsilon$ in Eq. (15) may alternatively be expressed according to Baillie et al. [2, Eq. (35) except $N = 8$ is used instead of $N = 7$ and the typographical error "64" removed] as

$$\epsilon = \pi^8 \left[ \phi_8\left(\frac{8}{2\pi}\right) - \phi_8\left(\frac{8-2\pi}{2\pi}\right) \right]. \tag{23}$$

Using the standard identity related to Bernoulli polynomials

$$\phi_N(x) - \phi_N(x-1) = \frac{(x-1)^{N-1}}{(N-1)!}\,,$$

$\epsilon$ in Eq. (23) can be written as

$$\epsilon = \frac{\pi^8}{(7)!}\left(\frac{4}{\pi} - 1\right)^7 = \frac{\pi\,(4-\pi)^7}{5040}$$

which is identical to Eq. (20). Therefore, the $2\pi$ congruence in the argument's numerator of the Bernoulli polynomials that was used by Baillie and the Borweins to justify this sudden break in $S_8$ may alternatively be explained by the aliasing phenomenon previously illustrated by the extra-bold-highlighted region of Figure 3.

As a final reference to the reports of Baillie and the Borweins [2, Example 3, Eq. (3)], we turn our attention back to Eq. (4) in order to relate the integral of sincs to their sums in a simple and intuitive manner. We now drop the periodicity condition on the previous signals (i.e., $p_i(t) = s_i(t)$ for all $t$). In this case, there is no constraint on $\alpha$. We obtain the Fourier transforms of $s_i(t)$ that turn out to be identical to the Fourier series coefficients found earlier [6] except that the index "$n$" is replaced with the continuous frequency variable "$f$". Applying the continuous form of Parseval's theorem [6], we obtain very analogous results related to the sinc integral, namely, if we define

$$\hat{I}_N(\alpha) = \int_0^{\infty} \text{sinc}^N(\alpha\pi f)df$$

then $\hat{I}_2(\alpha)$, $\hat{I}_4(\alpha)$, $\hat{I}_6(\alpha)$, and $\hat{I}_8(\alpha)$, respectively evaluate to $1/2\alpha$, $1/3\alpha$, $11/40\alpha$, and $151/630\,\alpha$ leading to the conclusion (refer to Eqs. (11)–(14)) that

$$\hat{S}_N(\alpha) = \hat{I}_N(\alpha) - \frac{1}{2} \quad \text{for } \alpha \leq 2/N \text{ and } N = 2, 4, 6, 8\ . \tag{24}$$

The process of successive convolutions and application of Parseval's theorem may thus be continued leading to the conclusion that $\hat{I}_N(\alpha)$ is a rational multiple of $1/\alpha$ (irrespective of non-zero $\alpha$) for all non-zero positive even integers N. This implies that $I_N$ in Eq. (3) (equal to $\hat{I}_N(1/\pi)$) is also a rational multiple of $\pi$ for all non-zero positive even integers $N$. In fact, since $\hat{S}_N(\alpha)$ is only defined for the non-aliasing range $\alpha \leq 2/N$, Eq. (24) also holds true for all non-zero positive even integers $N$. The problem lies with $S_N$ which is equal to $\hat{S}_N(1/\pi)$ only for $N = 2, 4, 6$ but not for $N = 8$ because $1/\pi > 1/4$. For this reason $S_N = I_N - 1/2$ for $N = 2, 4, 6$, but breaks into a polynomial in $\pi$ for $N \geq 8$ .

## 3 Conclusion

In this work, we presented a simple illustrated explanation of some remarkable results previously reported about sinc sums and integrals. We approached the problem with a signal processing background frequently utilizing tools such as convolution, Fourier series expansion, and Parseval's theorem. We illustrated the sudden break in sinc sums through the aliasing effect due to the horizontal smearing of a repeated self-convolution of a rectangle. Even though our illustrated approach was applied to only the first four even powers of sinc, we hope that it provided yet another simplistic and intuitively appealing explanation of this remarkable sinc behavior.

**Acknowledgement**

The author would like to thank Norbert Hungerbühler and the anonymous reviewers for their valuable comments that helped in improving the write-up and presentation of this paper.

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