Elem. Math. 72 (2017) 35 – 38
0013-6018/17/010035-4
DOI 10.4171/EM/321

Elemente der Mathematik

Short note From Pappus to Kocik's diagram for relativistic velocity addition

Gerhard Wanner

Recently, J. Kocik [4] discovered a nice diagram for Poincaré's formula for relativistic velocity addition

$$w = \frac{u+v}{1+uv} \tag{1}$$

(see Fig. 3, right). A. Sasane and V. Ufnarovski [7] gave three alternative geometric proofs. The purpose of this note is to show that still another proof is closely related to Carnot's solution of an ancient problem of Pappus (Prop. VII.117 in [6]).

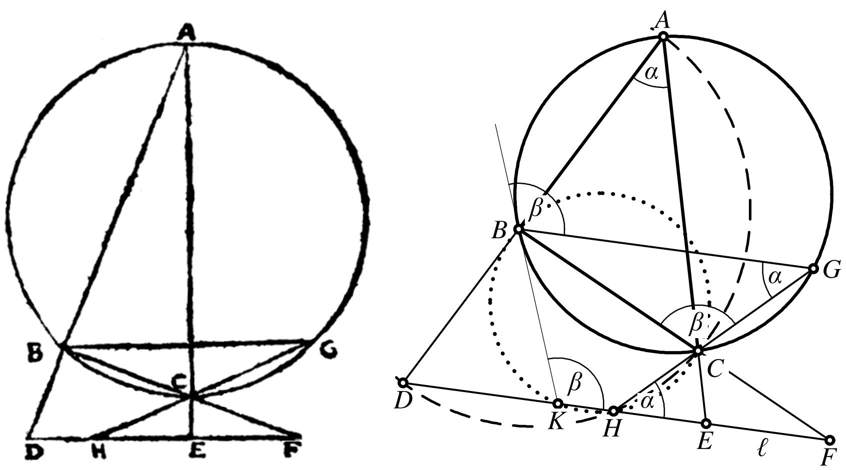

Figure 1 Pappus' problem VII.117 (from the 1660 edition, left), Pappus' solution and proof (right)

Pappus' problem. *For a given circle and three given points DEF located on a straight line ℓ ("tribus punctis DEF in recta linea"), find a triangle ABC inscribed in this circle whose sides (possibly extended) pass through D, E, F (see Fig. 1).*

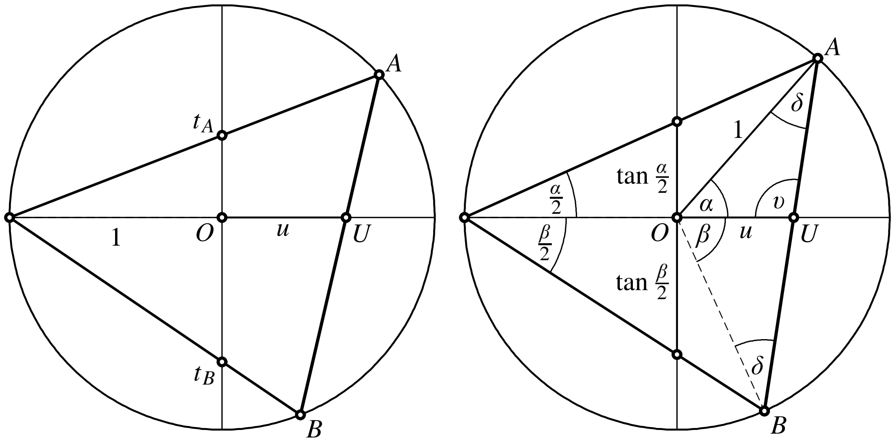

Figure 2 Carnot's formula (2) for the map $A \mapsto B$ through U

Pappus' solution. The points A, B, C are unknown, but we know the products $AE \cdot CE$ as well as $BF \cdot CF$ (Eucl. III.36)[1]. These quantities were later called *power of E and F with respect to this circle* by Steiner[2]. We now suppose the points A, B, C to be known and draw the chord BG parallel to ℓ and the line GCH with H on ℓ. We also draw the tangent BK with K on ℓ. Then (see Fig. 1, right)

$$\alpha \text{ at } A = \alpha \text{ at } G \text{ (Eucl. III.21)} = \alpha \text{ at } H \text{ (Eucl. I.29)} \Rightarrow ACHD \text{ cocyclic}$$

(the dashed circle, inverse of Eucl. III.22). Similarly,

$$\beta \text{ at } C = \beta \text{ at } B \text{ (Eucl. III.32)} = \beta \text{ at } K \text{ (Eucl. I.30)} \Rightarrow BCHK \text{ cocyclic}$$

(the dotted circle, again by Eucl. III.22). Now Eucl. III.36 applied to the dashed circle gives $DE \cdot HE = AE \cdot CE$, which determines the point H; Eucl. III.36 applied to the dotted circle gives $KF \cdot HF = BF \cdot CF$, which determines K (this last step is not drawn in Pappus' picture of Fig. 1, left, because it was the subject of his earlier Proposition VII.105). Finally the tangent from K gives B, the line BF gives C and the line EC gives A.

This beautiful proof remained standard for 15 centuries, until during the XVIII[th] century mathematicians (G. Cramer, L. Euler, J. Castillon, J.L. Lagrange) struggled to find simpler proofs, extensions to points DEF in arbitrary position, or more than three such points. Eventually, L.N.M. Carnot [3] in 1803 found an elegant solution for all cases.

Carnot's solution of the general problem. The main idea of Carnot was the use of $t_A = \tan\frac{\alpha}{2}$ and $t_B = -\tan\frac{\beta}{2}$ as "coordinates" for the points A and B on the unit circle (see Fig. 2). The use of the tangens theorem of Viète [8] applied to the triangle OUA gives

$$\frac{u-1}{u+1} = \frac{\tan\frac{\delta-\upsilon}{2}}{\tan\frac{\delta+\upsilon}{2}} = -\tan\frac{\beta}{2}\cdot\tan\frac{\alpha}{2} \qquad \text{or} \qquad t_B = \frac{u-1}{u+1}\cdot\frac{1}{t_A} \tag{2}$$

[1] All cited theorems Eucl. III.36, Eucl. III.21 etc. are known from high school, but perhaps not under these names.
[2] see, e.g., [5], p. 98.

because, by adding up the angles of the triangles OUA and OBA, we obtain, with Eucl. I.32, $\frac{\beta}{2} = \frac{v-\delta}{2}$ and $\frac{\alpha}{2} = 90^\circ - \frac{\delta+v}{2}$.

In the case where U does not lie on the x-axis, a little bit more complicated proof leads to a so-called Möbius transform between t_A and t_B. The group property of these transforms, discovered by Carnot when Möbius was still a boy, allowed him to solve any Pappus-like problem with arbitrary many points (see, e.g., [9]).

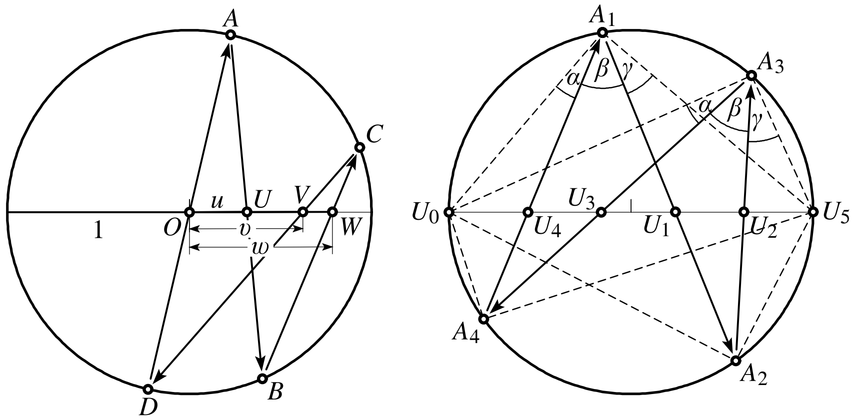

Figure 3 Kocik's diagram (left), generalization (right)

Proof of Kocik's diagram. We use here the right-hand formula of (2) for evaluating the map $A \mapsto B \mapsto C \mapsto D \mapsto A$ (see Fig. 3, left) with tangents $t_A \mapsto t_B \mapsto t_C \mapsto t_D \mapsto t_A$ through the points U, W, V, O and obtain by applying (2) repeatedly

$$t_A = \frac{-1}{1}\,\frac{1}{t_D} = \frac{-1}{1}\,\frac{v+1}{v-1}\,t_C = \cdots = \frac{-1}{1}\,\frac{v+1}{v-1}\,\frac{w-1}{w+1}\,\frac{u+1}{u-1}\,t_A\,. \tag{3}$$

This map returns for any initial point A back to A exactly if

$$(v-1)(w+1)(u-1) = -(v+1)(w-1)(u+1) \tag{4}$$

which, when solved for w, gives equation (1). The particular case where A lies at the North Pole is Kocik's original diagram.

Proof with projective geometry. We move O to an arbitrary position U_4 with coordinate u_4, so that equations (3) and (4) become (see Fig. 3, right)

$$\frac{u_4-1}{u_4+1}\,\frac{u_3+1}{u_3-1}\,\frac{u_2-1}{u_2+1}\,\frac{u_1+1}{u_1-1} = 1 \tag{5}$$

which, with $-1 = u_0$ and $1 = u_5$, can be written as

$$\frac{u_4-u_5}{u_4-u_0} : \frac{u_1-u_5}{u_1-u_0} = \frac{u_3-u_5}{u_3-u_0} : \frac{u_2-u_5}{u_2-u_0}\,. \tag{6}$$

This is another theorem of Pappus (VII.129), saying that under perspective projections the cross ratios (U_5, U_0, U_4, U_1) and (U_5, U_0, U_3, U_2) are the same. This general case makes thus the third proof of [7] much simpler. Relation (5) (in a different notation) is due to A.L. Candy [2] and the elegant proof via Pappus' theorem to L. Bankoff [1].

Direct trigonometric proof. We connect in Fig. 2 A with the "South-Pole" and B with the "North-Pole" and create so Kocik's original diagram. The angles $90° - \alpha$ and $90° - \beta$ then lead to the angles $45° - \frac{\alpha}{2}$ and $45° - \frac{\alpha}{2}$ at the periphery (Eucl. III.20). Kocik's u and v then become (remember $\tan 45° = 1$)

$$u = \tan\Big(45° - \frac{\alpha}{2}\Big) = \frac{1-\tan\frac{\alpha}{2}}{1+\tan\frac{\alpha}{2}} \;\Rightarrow\; \tan\frac{\alpha}{2} = \frac{1-u}{1+u}\,, \text{ also } \tan\frac{\beta}{2} = \frac{1-v}{1+v}\,, \tag{7}$$

so that Viète's formula in (2) (with u replaced by w) is

$$\frac{1-w}{1+w} = \tan\frac{\beta}{2}\cdot\tan\frac{\alpha}{2} = \frac{1-u}{1+u}\cdot\frac{1-v}{1+v}\,, \tag{8}$$

another nice way to write (4) and thus (1).

Remark. The author wishes to acknowledge helpful hints from F. Sigrist (Neuchâtel) and Djura Paunić (Novi Sad).

References

[1] L. Bankoff, *The metamorphosis of the Butterfly problem*, Mathematics Magazine **60**, (1987), pp. 195–210.

[2] A.L. Candy, *A general theorem relating to transversals, and its consequences*, Annals of Math. **10**, (1896), pp. 175–190; **11**, pp. 10–19.

[3] L.N.M. Carnot, *Géométrie de position*, Duprat, Paris 1803.

[4] J. Kocik, *Geometric diagram for relativistic addition of velocities*, Amer. J. of Physics **80**, (2012) pp. 737–739.

[5] A. Ostermann, G. Wanner, *Geometry by its History*, Springer (2012).

[6] Pappus of Alexandria (appr. 300 A.D.), *Collection* (or $\sigma\upsilon\nu\alpha\gamma\omega\gamma\acute{\eta}$), first Latin translation by F. Commandino (published in 1588 and 1660); definite Greek-Latin edition by F. Hultsch, Berolini 1876–1878; French translation: Ver Eecke (1933).

[7] A. Sasane, V. Ufnarovski, *Alternative proofs for Kocik's geometric diagram for relativistic velocity addition*, Elemente der Mathematik, **71** (2016), pp. 122–130.

[8] F. Viète, *Variorum de rebus Mathematicis Responsorum Liber VIII*, Tours 1593; see van Schooten (1646), *Opera Mathematica*, pp. 347–436.

[9] G. Wanner, *The Cramer–Castillon problem and Urquart's 'most elementary' theorem*, Elemente der Mathematik, **61** (2006), pp. 58–64.

Gerhard Wanner
Univ. Geneva
e-mail: Gerhard.Wanner@unige.ch