Elem. Math. 72 (2017) 62 – 65
0013-6018/17/020062-4
DOI 10.4171/EM/325

Elemente der Mathematik

A note on Sassenfeld matrices

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In 1951, H. Sassenfeld published a paper [3] (in German) in the context of iterative numerical methods for the solution of linear systems of equations. As part of his work, he introduced a class of matrices, subsequently referred to as *Sassenfeld matrices*, for which he was able to establish convergence of the famous Gauß–Seidel iteration scheme (see, e.g., [1, §11.2]). Since that time Sassenfeld's convergence criterion has found its way into various texts in numerical mathematics. The goal of this note is to present a short discussion about Sassenfeld matrices in the context of the Gauß–Seidel method, and, in particular, to provide an elementary proof of the fact that such matrices are invertible.

To set the stage, for $m \geq 2$, we consider an $m \times m$ square matrix $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, with nonvanishing diagonal entries. Denoting, for $1 \leq i, j \leq m$, by a_{ij} the entry of $\boldsymbol{A}$ in row i and

Bei der Lösung von mathematischen Problemen in Anwendungen treten oftmals riesige lineare Gleichungssysteme mit Millionen von Unbekannten auf. Selbst mithilfe von Rechnerunterstützung ist die Komplexität des klassischen Gaußschen Eliminationsverfahrens dann meistens viel zu gross, um eine rasche Lösung zu erlauben. Alternativ werden iterative numerische Methoden, zu denen auch das berühmte Gauß–Seidel Verfahren gehört, erfolgreich eingesetzt. In der vorliegenden Arbeit erinnern die Autoren an ein hinreichendes Konvergenzkriterium für die Gauß–Seidel Iteration von Helmut Sassenfeld aus dem Jahr 1951 und machen einige ergänzende Beobachtungen.

column j (analogous notation is used for other matrices), and by

$$l_{ij} = \begin{cases} a_{ij} & \text{if } i > j \\ 0 & \text{otherwise} \end{cases}, \qquad d_{ij} = \begin{cases} a_{ii} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \qquad u_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{otherwise} \end{cases},$$

the corresponding entries below, on, and above the diagonal of $\boldsymbol{A}$, respectively, we obtain the splitting $a_{ij} = l_{ij} + d_{ij} + u_{ij}$, or, equivalently, in matrix form $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{D} + \boldsymbol{U}$. Then, if $\boldsymbol{A}$ is invertible, the Gauß–Seidel method for solving a linear system of equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

where $\boldsymbol{b} \in \mathbb{R}^m$ is a given right-hand side vector, is given iteratively by the lower triangular system

$$(\boldsymbol{D} + \boldsymbol{L})\boldsymbol{x}_n = -\boldsymbol{U}\boldsymbol{x}_{n-1} + \boldsymbol{b}, \qquad n \geq 1, \tag{2}$$

with an initial guess $\boldsymbol{x}_0 \in \mathbb{R}^m$. Since the diagonal entries of $\boldsymbol{A}$ are nonzero, we note that the above system (2) is uniquely solvable for any $n \geq 1$.

In order to discuss the convergence of the iterates $\{\boldsymbol{x}_n\}_{n\geq 0}$ from (2) to the exact solution $\boldsymbol{x}$ of the original linear system (1), we define a sequence of numbers $s_1, \ldots, s_m$ by means of the recursive relation

$$s_i = \frac{1}{|a_{ii}|}\left(\sum_{1\leq j<i} s_j|a_{ij}| + \sum_{i<j\leq m} |a_{ij}|\right) = \frac{1}{|d_{ii}|}\left(\sum_{1\leq j<i} s_j|l_{ij}| + \sum_{i<j\leq m} |u_{ij}|\right),$$

for $i = 1, \ldots, m$. We call $\mu(\boldsymbol{A}) = \max_{1\leq i\leq m} s_i \geq 0$ the *Sassenfeld index* of $\boldsymbol{A}$. Furthermore, we say that $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ is a *Sassenfeld matrix* if $\mu(\boldsymbol{A}) < 1$.

Before stating our first result, we define the ∞-norm of a vector $\boldsymbol{v} \in \mathbb{R}^m$ with components $\{v_i\}_{i=1}^m$ by $\|\boldsymbol{v}\|_\infty = \max_{1\leq i\leq m} |v_i|$. Furthermore, for any matrix $\boldsymbol{M} \in \mathbb{R}^{m\times m}$ the induced matrix norm is given by $\|\boldsymbol{M}\|_\infty = \sup_{\boldsymbol{v}\in\mathbb{R}^m, \|\boldsymbol{v}\|_\infty=1} \|\boldsymbol{M}\boldsymbol{v}\|_\infty$. We notice the estimate $\|\boldsymbol{M}\boldsymbol{v}\|_\infty \leq \|\boldsymbol{M}\|_\infty\|\boldsymbol{v}\|_\infty$.

Proposition 1. *Within the above framework there holds the bound*

$$\|(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\|_\infty \leq \mu(\boldsymbol{A}).$$

Proof. Consider an arbitrary vector $\boldsymbol{v} \in \mathbb{R}^m$ with $\|\boldsymbol{v}\|_\infty = 1$, and define

$$\boldsymbol{w} = (\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\boldsymbol{v}.$$

Then, there holds $(\boldsymbol{D} + \boldsymbol{L})\boldsymbol{w} = \boldsymbol{U}\boldsymbol{v}$, and, thus, $\boldsymbol{w} = \boldsymbol{D}^{-1}(-\boldsymbol{L}\boldsymbol{w} + \boldsymbol{U}\boldsymbol{v})$. Writing this equation componentwise, we have that

$$w_i = \frac{1}{a_{ii}}\left(-\sum_{1\leq j<i} a_{ij}w_j + \sum_{i<j\leq m} a_{ij}v_j\right), \qquad 1 \leq i \leq m.$$

Then, taking moduli on either side of the above equality, and recalling that $\|\boldsymbol{v}\|_\infty = 1$, leads to

$$|w_i| = \frac{1}{|a_{ii}|}\left|-\sum_{1\leq j<i} a_{ij}w_j + \sum_{i<j\leq m} a_{ij}v_j\right| \leq \frac{1}{|a_{ii}|}\left(\sum_{1\leq j<i} |a_{ij}||w_j| + \sum_{i<j\leq m} |a_{ij}|\right).$$

For $i = 1$, this implies that $|w_1| \leq s_1 \leq \mu(\boldsymbol{A})$.

Furthermore, if, for some $i \in \{2, \ldots, m\}$, there holds that $|w_j| \leq s_j$, for all $j = 1, \ldots, i-1$, then we derive the bound

$$|w_i| \leq \frac{1}{|a_{ii}|}\left(\sum_{1\leq j<i} |a_{ij}|s_j + \sum_{i<j\leq m} |a_{ij}| \right) = s_i \leq \mu(\boldsymbol{A}).$$

By induction, we conclude that $|w_i| \leq \mu(\boldsymbol{A})$ for any $i = 1, \ldots, m$. Therefore, we infer that $\|\boldsymbol{w}\|_\infty \leq \mu(\boldsymbol{A})$, and the proof is complete. □

The following two results follow immediately from the above proposition.

Theorem 2. *If a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, with $m \geq 2$, or its transpose, $\boldsymbol{A}^\top$, is a Sassenfeld matrix, then $\boldsymbol{A}$ is invertible.*

Proof. Suppose that $\boldsymbol{A}$ is a non-invertible Sassenfeld matrix. Then, there exists a vector $\boldsymbol{v}^\star \in \mathbb{R}^m$, with $\|\boldsymbol{v}^\star\|_\infty = 1$, such that $\boldsymbol{A}\boldsymbol{v}^\star = \boldsymbol{0} \in \mathbb{R}^m$. This implies that

$$(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\boldsymbol{v}^\star = -\boldsymbol{v}^\star.$$

Hence, by means of Proposition 1, we conclude that

$$\mu(\boldsymbol{A}) \geq \|(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\|_\infty \geq \|(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\boldsymbol{v}^\star\|_\infty = \|\boldsymbol{v}^\star\|_\infty = 1,$$

which constitutes a contradiction to the assumption $\mu(\boldsymbol{A}) < 1$. We close the proof by remarking that $\boldsymbol{A}$ is invertible if and only if $\boldsymbol{A}^\top$ is invertible. □

Remark 3. We note that Theorem 2 is sharp in the sense that there are non-invertible matrices $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ with $\mu(\boldsymbol{A}) = 1$. To see this, for $\alpha \geq 1$, let us consider the matrix

$$\boldsymbol{A}_\alpha = \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \\ 1 & 0 & \cdots & 0 & \alpha \end{pmatrix} \in \mathbb{R}^{m\times m},$$

for which there holds that $\mu(\boldsymbol{A}_\alpha) = \alpha \geq 1$ (and, in particular, $\mu(\boldsymbol{A}_1) = 1$). Furthermore, for $\alpha \to \infty$, this example illustrates that non-invertible matrices may have arbitrarily large Sassenfeld indices; incidentally, replacing the entry 1 in the bottom left corner of $\boldsymbol{A}_\alpha$ by 0, we see that the same observation applies to invertible matrices. Finally, we remark that a matrix features Sassenfeld index 0 if and only if it is lower triangular (with non-vanishing diagonal entries). ◇

Theorem 3. *If $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ is a Sassenfeld matrix, then the Gauß–Seidel scheme* (2) *converges to the unique solution $\boldsymbol{x}$ of* (1) *for any starting vector $\boldsymbol{x}_0 \in \mathbb{R}^m$.*

Proof. For $n \geq 0$ we signify by $\boldsymbol{e}_n = \boldsymbol{x} - \boldsymbol{x}_n$ the difference between the exact solution $\boldsymbol{x}$ of (1) and the nth iterate $\boldsymbol{x}_n$ of the Gauß–Seidel iteration (2). Then, for $n \geq 1$, we observe the identities

$$\begin{aligned} \boldsymbol{x} &= -(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\boldsymbol{x} + (\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{b}, \\ \boldsymbol{x}_n &= -(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\boldsymbol{x}_{n-1} + (\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{b}. \end{aligned}$$

Taking the difference yields $\boldsymbol{e}_n = -(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\boldsymbol{e}_{n-1}$. Therefore, due to Proposition 1, we arrive at

$$\|\boldsymbol{e}_n\|_\infty = \|(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\boldsymbol{e}_{n-1}\|_\infty \leq \|(\boldsymbol{D} + \boldsymbol{L})^{-1}\boldsymbol{U}\|_\infty \|\boldsymbol{e}_{n-1}\|_\infty \leq \mu(\boldsymbol{A})\|\boldsymbol{e}_{n-1}\|_\infty.$$

Recalling that $\mu(\boldsymbol{A}) < 1$, and iterating the above bound, results in

$$\|\boldsymbol{e}_n\|_\infty \leq \mu(\boldsymbol{A})^n \|\boldsymbol{e}_0\|_\infty \to 0,$$

as $n \to \infty$. This shows the convergence of the Gauß–Seidel scheme (2). □

Remark 4. The above Theorem 3 provides a sufficient assumption for the convergence of the Gauß–Seidel iteration scheme (2). We emphasize, however, that this criterion is by no means necessary. For instance, for $\beta \geq 1$, consider the matrix $\boldsymbol{B}_\beta \in \mathbb{R}^{m \times m}$ with entries

$$(b_\beta)_{ij} = \begin{cases} 1 & \text{for } i = j \\ \beta & \text{for } i = 1, j = 2 \\ 0 & \text{otherwise} \end{cases}, \qquad 1 \leq i, j \leq m.$$

Here, it holds $\mu(\boldsymbol{B}_\beta) = \beta \geq 1$, and the Gauß–Seidel method converges in two steps for any right-hand side vector $\boldsymbol{b} \in \mathbb{R}^m$ and any initial guess $\boldsymbol{x}_0 \in \mathbb{R}^m$. ◇

In order to close this note we provide a simple (pseudocode) MATLAB [2] function for computing the Sassenfeld index $\mu(\boldsymbol{A})$ of a quadratic matrix with nonzero diagonal entries.

```
function μ = SassenfeldIndex(A)
    A = abs(A);                                  ▷ remove signs from all matrix entries
    [m, n] = size(A);
    if isempty(A) || ne(m, n) || nnz(~diag(A)) then   ▷ check whether matrix is admissible
        disp('matrix not admissible'); return ;
    end if
    s = ones(1, m);                              ▷ initialize Sassenfeld vector s = (s_1, ..., s_m)
    for i = 1 : m do
        s(i) = 0;
        s(i) = dot(s, A(i, :))/A(i, i);          ▷ compute Sassenfeld index of A
    end for
    μ = max(s);
end function
```

References

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