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An effective approach for integer partitions using exactly two distinct sizes of parts

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1 Introduction

A partition of a positive integer n is a sequence of non-increasing positive integers n_1 (a_1 times), n_2 (a_2 times), ..., n_s (a_s times), with $n_i > n_{i+1}$, that sum to n. We sometimes write such partition $\pi = (n_1^{a_1}\ n_2^{a_2} \cdots\ n_s^{a_s})$, each n_i is called part of the partition π and a_i its frequency. The partition function $p(n)$ counts the partitions of n. If we ignore some

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Partitionen von natürlichen Zahlen sind ein gut untersuchtes Gebiet in der kombinatorischen Zahlentheorie. Schon Euler gab die erzeugende Funktion für die Anzahl solcher Partitionen an. Von grossem Interesse sind auch Partitionen mit bestimmten Eigenschaften, etwa einer gegebenen Anzahl Summanden, mit vorgegebenem kleinsten Summanden usw. In der vorliegenden Arbeit werden Partitionen betrachtet, bei denen nur zwei verschiedene Summanden auftreten, also zum Beispiel $11 = 4+4+1+1+1$. Insbesondere finden die Autoren eine explizite Formel für die Anzahl solcher Partitionen. Diese Formel wird in einen schlanken Algorithmus gegossen, der diese Partitionen liefert. Als Nebenprodukt resultiert eine Formel für die Partitionen einer Zahl n, bei denen nur s teilerfremde Summanden auftreten.

unpublished work of G.W. Leibniz, the theory of integer partitions can find its origin in the work of L. Euler [6]. In fact, he made a sustained study of partitions and partition identities, and exploited them to establish a huge number of results in Analysis in 1748. An excellent introduction to this subject can be found in the book of G.E. Andrews [2].

Definition 1.1. Let $\pi = (n_1^{a_1}\, n_2^{a_2} \cdots n_s^{a_s})$ be a partition of n. We say that π is a partition into k parts with s distinct sizes if

$$\begin{cases} n = a_1\, n_1 + \cdots + a_s\, n_s; \\ n_1 > n_2 > \cdots > n_s \geq 1; \\ a_1 + \cdots + a_s = k; \\ a_1, \ldots, a_s \geq 1. \end{cases} \tag{1.1}$$

Let $t(n, k, s)$ be the number of solutions of system (1.1) and $t(n, s)$ the total number of partitions of n into s distinct sizes. Then we have

$$t(n, s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} t(n, k, s). \tag{1.2}$$

Example 1.2. Among 27 partitions of $n = 11$ into two distinct sizes, the partitions $(7^1\, 1^4)$, $(4^2\, 1^3)$, $(3^1\, 2^4)$ and $(3^3\, 1^2)$ are the only ones which are into 5 parts.

This kind of partitions appeared for the first time in the work of P.A. MacMahon [7]. Next, E. Deutsch presented the number of partitions of n into exactly two odd sizes of parts and the number of partitions of n into exactly two sizes of parts, one odd and one even. One can find these values in the Online Encyclopedia of Integer Sequences (OEIS) [8] as A117955 for the first number, A117956 for the second one and A002133 for the number of partitions of n using only 2 types of parts.

In [3] we can find a proof of effective and non-effective finiteness theorems on $t(n, k, s)$, we can cite for example the following results:

Theorem 1.3. *For $k \geq s \geq 2$ and $n \geq k + \frac{s(s-1)}{2}$, we have*

$$t(n, k, s) = \sum_{i=1}^{\lfloor \frac{2n-s(s-1)}{2k} \rfloor} \sum_{j=1}^{k-s+1} t(n - ki, k - j, s - 1), \tag{1.3}$$

$$t(n, k, 2) = \sum_{i=1}^{\lfloor \frac{n-1}{k} \rfloor} \tau_{k-1\downarrow}(n - ki), \tag{1.4}$$

where $\tau_{d\downarrow}(k)$ denotes the number of positive divisors of k less than or equal to d.

2 Main results

One of the aims of this paper is to give an explicit formula for $t(n, k, 2)$ using an effective new approach. Thus, let us consider the system:

$$\begin{cases} n = a_1\, n_1 + a_2\, n_2; \\ a_1 + a_2 = k; \\ n_1 > n_2 \geq 1; \\ a_1, a_2 \geq 1. \end{cases} \tag{2.1}$$

Throughout the remainder of the paper let $m = n_1 - n_2$ for any solution of system (2.1).

First of all, we introduce the following lemma to prepare the main theorem.

Lemma 2.1. *System* (2.1) *has integral solutions if and only if the following conditions are satisfied:*

(i) $n \equiv n_2 k \pmod{m}$,

(ii) $\max\left(1, \left\lceil \frac{n}{k} \right\rceil - m + \chi(k|n)\right) \leq n_2 \leq \left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n)$,

where $\chi(k|n) = 1$ *if k divides n, and* 0 *otherwise.*

Proof. By system (2.1) we have

$$\begin{pmatrix} n_1 & n_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} n \\ k \end{pmatrix}.$$

Since $m > 0$, we get

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} n - n_2 k \\ -n + n_1 k \end{pmatrix}.$$

Then, system (2.1) has integral solutions if and only if m divides $n - n_2 k$, $n - n_2 k > 0$ and $-n + n_1 k > 0$. That is,

$$n \equiv n_2 k \pmod{m} \quad \text{and} \quad \frac{n}{k} - m < n_2 < \frac{n}{k}.$$

Dependent on whether k divides n, plus or minus 1 needs to be added, which is done by $\chi(k|n)$ and the required result holds. □

From this lemma, we can now derive the following theorem.

Theorem 2.2. *For* $k \geq 2$ *and* $n \geq k + 1$ *let* $d = \gcd(n, k)$ *and for any divisor e of d let* $\Im_e$ *be the set of pairs* $(\alpha, \beta) \in \mathbb{N}^2$*, such that:*

- $1 \leq \alpha \leq \left\lfloor \frac{n-k}{e} \right\rfloor$ *and* $\gcd\left(\alpha, \frac{k}{e}\right) = 1$,
- $\beta \equiv \left(\frac{n}{e}\right)\left(\frac{k}{e}\right)^{-1} \pmod{\alpha}$ *and* $0 \leq \beta \leq \min\left(\alpha - 1, \left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n)\right)$.

Then

$$t(n,k,2) = \sum_{e|d} \sum_{(\alpha,\beta)\in\Im_e} \left(\left\lfloor \frac{\lfloor \frac{n}{k} \rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor - \left\lceil \frac{\max\left(1, \lceil \frac{n}{k} \rceil + \chi(k|n) - \alpha e\right) - \beta}{\alpha} \right\rceil + 1 \right).$$

Proof. Put $e = \gcd(m,k)$ and let $\alpha = \frac{m}{e}$, that is $1 \le \alpha \le \lfloor \frac{n-k}{e} \rfloor$ and $\gcd\left(\alpha, \frac{k}{e}\right) = 1$. By Lemma 2.1, case (i), we can see that e divides d and $n_2 \equiv \left(\frac{n}{e}\right)\left(\frac{k}{e}\right)^{-1} \pmod{\alpha}$.

Let $0 \le \beta < \alpha$, such that $\beta \equiv \left(\frac{n}{e}\right)\left(\frac{k}{e}\right)^{-1} \pmod{\alpha}$. Then

$$n_2 = \beta + t\alpha, \ t \in \mathbb{Z}.$$

Since $\beta \le n_2$, then $t \in \mathbb{N}$. It follows from Lemma 2.1, case (ii), that

$$\max\left(1, \left\lceil \frac{n}{k} \right\rceil - m + \chi(k|n)\right) \le \beta + t\alpha \le \left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n)\cdot$$

Finally, $t(n,k,2)$ equals the number of positive integers t, such that

$$\left\lceil \frac{\max\left(1, \lceil \frac{n}{k} \rceil - m + \chi(k|n)\right) - \beta}{\alpha} \right\rceil \le t \le \left\lfloor \frac{\lfloor \frac{n}{k} \rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor.$$

This completes the proof. □

Remark 2.3. One nice application of Theorem 2.2 concerns Algorithm 1 which allows to generate all partitions of n using exactly two distinct sizes of parts.

We remark that Algorithm 1 runs in $O(n)$.

Example 2.4. Let $n = 22$ and $k = 8$, then $d = \gcd(22, 8) = 2$. So, we have two divisors of d, $e = 1$ and $e = 2$.

Case 1: $e = 1$.
The values of α that satisfy $1 \le \alpha \le 14$ and $\gcd(\alpha, 8) = 1$ are $1, 3, 5, 7, 9, 11$ or 13.

1. For $\alpha = 1$, we get $\beta = 0$. The pair $(1, 0)$ is accepted and gives the values:

$$t = 1, \ n_2 = 2, \ n_1 = 3, \ a_2 = 2 \text{ and } a_1 = 6,$$

and then the partition $(3^6\ 2^2)$.

2. For $\alpha = 3$, we get $\beta = 2$. The pair $(3, 2)$ is accepted and gives the values:

$$t = 1, \ n_2 = 2, \ n_1 = 5, \ a_2 = 6 \text{ and } a_1 = 2,$$

and then the partition $(5^2\ 2^6)$.

Algorithm 1. Partitions into k parts with exactly two distinct sizes of parts.

Require: $k \geq 2, n \geq \max\{k, 3\}$
Ensure: Set of quadruple (n_1, a_1, n_2, a_2),

```
d ← gcd(n, k)
for each divisor e of d do
  for α from 1 to ⌊(n−k)/e⌋ do
    if gcd(α, k/e) = 1 then
      β ← (n/e)(k/e)^{-1} (mod α)
      if β ≤ min(α − 1, ⌊n/k⌋ − χ(k|n)) then
        t1 ← ⌈ max(1, ⌈n/k⌉ − αe + χ(k|n)) − β ) / α ⌉
        t2 ← ⌊ (⌊n/k⌋ − χ(k|n) − β) / α ⌋
        for t from t1 to t2 do
          n2 ← β + tα
          n1 ← αe + n2
          a2 ← ⌊ (n − n1 k) / (n2 − n1) ⌋
          a1 ← k − a2
        end for
      end if
    end if
  end for
end for
```

3. For $\alpha = 5$, we get $\beta = 4 > \min(4, 2)$, then the pair $(5, 4)$ is rejected.
4. For $\alpha = 7$, we get $\beta = 1$. The pair $(7, 1)$ is accepted and gives the values:

$$t = 1,\ n_2 = 1,\ n_1 = 8,\ a_2 = 6 \text{ and } a_1 = 2,$$

   and then the partition $(8^2\ 1^6)$.
5. For $\alpha = 9$, we get $\beta = 5 > \min(8, 2)$, then the pair $(9, 5)$ is rejected.
6. For $\alpha = 11$, we get $\beta = 3 > \min(10, 2)$, then the pair $(11, 3)$ is rejected.
7. For $\alpha = 13$, we have $\beta = 6 > \min(13, 2)$, then the pair $(13, 6)$ is rejected.

Case 2: $e = 2$.
The values of α that satisfy $1 \leq \alpha \leq 7$ and $\gcd(\alpha, 8) = 1$ are 1, 3, 5 or 7.

1. For $\alpha = 1$, we have $\beta = 0$. The pair $(1, 0)$ is accepted and gives $1 \leq t \leq 2$. Applying Algorithm 1, we obtain two partitions corresponding to the pair $(1, 0)$; the first one is $(3^7\ 1^1)$ for $t = 1$ and the second one is $(4^3\ 2^5)$ for $t = 2$.

2. For $\alpha = 3$, we get $\beta = 2$. The pair (3, 2) is accepted and gives the values:

$$t = 1,\ n_2 = 2,\ n_1 = 8,\ a_2 = 7 \text{ and } a_1 = 1,$$

and then the partition $(8^1\ 2^7)$.

3. For $\alpha = 5$, we get $\beta = 4 > \min(4, 2)$, the pair (5, 4) is rejected.
4. For $\alpha = 7$, we get $\beta = 1$. The pair (7, 1) is accepted and gives the values:

$$t = 1,\ n_2 = 1,\ n_1 = 15,\ a_2 = 7 \text{ and } a_1 = 1,$$

and then the partition $(15^1\ 1^7)$.

We get, finally,

$$t(22, 8, 2) = 7.$$

3 Partitions into distinct co-prime sizes

After having counted the number $t(n, k, s)$, it would be of considerable interest to explore the number of partitions of n into k parts with exactly s distinct co-prime sizes, which we denote by $g(n, k, s)$. Thus, let us set

$$g(n, s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} g(n, k, s). \tag{3.1}$$

Theorem 3.1. *For $k \geq s \geq 2$ and $n \geq k + \frac{s(s-1)}{2}$, we have*

$$g(n, k, s) = \sum_{d|n} \mu\left(\frac{n}{d}\right) t(d, k, s), \tag{3.2}$$

where $\mu(.)$ denotes the Möbius function.

Proof. Let $T(n, k, s)$ be the set of partitions of n into k parts with s distinct sizes and $G(n, k, s)$ the subset of such partitions but with s distinct co-prime sizes. We notice that, the mapping from the set $T(n, k, s)$ to $\bigcup_{d|n} G(d, k, s)$ defined by:

$$(n_1^{a_1}\ n_2^{a_2} \cdots n_s^{a_s}) \to \left(\left(\frac{n_1}{\delta}\right)^{a_1} \left(\frac{n_2}{\delta}\right)^{a_2} \cdots \left(\frac{n_s}{\delta}\right)^{a_s}\right),$$

is a bijection, where $\delta = \gcd(n_1, n_2, \ldots, n_s)$.

Consequently, we have

$$t(n, k, s) = \sum_{d|n} g(d, k, s). \tag{3.3}$$

Hence, the result follows by using the Möbius inversion formula. □

Remark 3.2. Since $t(d, k, s) = 0$ if $d < k + \frac{s(s-1)}{2}$, the summation in (3.2) can be extended only over all divisors d of n such that $d \geq k + \frac{s(s-1)}{2}$. For example, if we take $n = 22$ and $k = 8$, then

$$g(22, 8, 2) = \mu(2)\, t(11, 8, 2) + \mu(1)\, t(22, 8, 2).$$

It can be checked using Algorithm 1 that if $n = 11$ and $k = 8$, then $t(11, 8, 2) = 2$, such partitions are $(2^3\ 1^5)$ and $(4^1\ 1^7)$. Then, according to Example 2.4, we get $g(22, 8, 2) = 7 - 2 = 5$. The partitions in question are: $(3^7\ 1^1)$, $(3^6\ 2^2)$, $(5^2\ 2^6)$, $(8^2\ 1^6)$ and $(15^1\ 1^7)$.

Using Theorems 3.1 and 2.2, we can construct Table 1.

Table 1. $g(n, k, 2)$, $2 \leq k < n \leq 20$.

$n\backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$g(n, 2)$
3	1																		1
4	1	1																	2
5	2	2	1																5
6	1	1	2	1															5
7	3	3	2	2	1														11
8	2	2	2	2	2	1													11
9	3	3	2	3	2	2	1												16
10	2	2	4	1	3	2	2	1											17
11	5	5	3	4	2	3	2	2	1										27
12	2	2	2	2	3	2	3	2	2	1									21
13	6	6	4	5	2	4	2	3	2	2	1								37
14	3	3	5	3	4	1	4	2	3	2	2	1							33
15	4	4	3	3	4	4	2	4	2	3	2	2	1						38
16	4	4	5	3	4	3	3	2	4	2	3	2	2	1					42
17	8	8	5	7	3	5	3	4	2	4	2	3	2	2	1				59
18	3	3	5	2	5	2	4	2	4	2	4	2	3	2	2	1			46
19	9	9	6	7	3	7	3	4	3	4	2	4	2	3	2	2	1		71
20	4	4	4	4	4	3	6	2	3	3	4	2	4	2	3	2	2	1	57

From identity (3.2) we can see that if $k \geq \lfloor \frac{n}{2} \rfloor$, then $t(n, k, 2) = g(n, k, 2)$. In the present theorem we present this observation in a more explicit form.

Theorem 3.3. *For $n \geq k + 1$ and $k \geq \max\{2, \lfloor \frac{n}{2} \rfloor\}$, we have*

$$t(n, k, 2) = g(n, k, 2) = \tau(n - k) - \chi(n),$$

where $\tau(n)$ *denotes the number of positive divisors of* n *and* $\chi(n) = 1$ *if* $n = 2k$, 0 *otherwise.*

Proof. Let us first notice that if $k \geq 1+\max\{2, \lfloor \frac{n}{2} \rfloor\}$, then $k \geq \lceil \frac{n+1}{2} \rceil$, and by Identity (1.4) the result yields (see [3], Corollary 3). Let now $k = \max\{2, \lfloor \frac{n}{2} \rfloor\}$. Since the result is true for $n = 3$, we can assume $k = \lfloor \frac{n}{2} \rfloor$. Let $\pi = (n_1^{a_1}\ n_2^{a_2})$ be a partition of n into k parts with two distinct sizes. If n is even, then $n_2 = 1$, else $n > (a_1 + a_2)\, n_2 = k\, n_2 \geq 2 \left\lfloor \frac{n}{2} \right\rfloor = n$, a contradiction. Hence, $n - k = (n_1 - 1)\, a_1$, in which case $n_1 - 1$ divides $n - k$. So, for each divisor d of $n - k$, we get $n_1 = d + 1$, $a_1 = \frac{n-k}{d} > 0$ and $a_2 = k - \frac{n-k}{d} > 0$, except for $d = 1$, where $a_2 = k - \frac{n-k}{d} = 0$. Thus, the result follows.

Now, if n is odd, then $n_2 = 1$ or $(n_1, n_2) = (3, 2)$. Indeed, if ($n_2 = 2$ and $n_1 \geq 4$) or ($n_2 \geq 3$), then $n > 3a_1 + 2a_2 = 2k + a_1 \geq 2 \left\lfloor \frac{n}{2} \right\rfloor + 1 = n$, a contradiction. In case of $n_2 = 1$, by the same argument as above, we get for each divisor d of $n - k$, $n_1 = d + 1$, $a_1 = \frac{n-k}{d} > 0$ and $a_2 = k - \frac{n-k}{d} > 0$, except for $d = 1$, where $a_2 = k - \frac{n-k}{d} < 0$, which is completed by the partition $(3^{n-2k}\ 2^{3k-n})$. This completes the proof. □

Remark 3.4. As shown in the proof above, the $t(n, k, 2)$ partitions have been generated explicitly, in fact, for each divisor d of $n - k$, we have :

$$(n_1^{a_1}\ n_2^{a_2}) = \begin{cases} \left((d+1)^{\frac{n-k}{d}}\ \ 1^{k-\frac{n-k}{d}}\right) & \text{if } \left(k > \lfloor \frac{n}{2} \rfloor\right) \text{ or } \left(k = \lfloor \frac{n}{2} \rfloor \text{ and } d \neq 1\right); \\ (3^{n-2k}\ \ 2^{3k-n}) & \text{if } n \text{ odd, } k = \lfloor \frac{n}{2} \rfloor \text{ and } d = 1; \\ \text{Does not exist} & \text{if } n \text{ even, } k = \frac{n}{2} \text{ and } d = 1. \end{cases}$$

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