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Triangles with sides in arithmetic progression

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In this note we study geometric properties of triangles $\triangle ABC$ the sides of which are in arithmetic progression, $CA - BC = AB - CA$. As far as the authors know, this topic does not seem to have been in the spotlight[1]. This is why we would like to integrate some known results and (possibly) new results here.

Theorem 1. *Let $\triangle ABC$ be a triangle with edge lengths $a = BC, b = CA, c = AB$. Then, each of the following three conditions is a necessary and sufficient condition so that the sequence of sides a, b, c is an arithmetic progression.*

(i) *The line joining the centroid G and the incenter I is parallel to the edge AC.*

[1] with the exception of the study in the case of Heronian triangles, i.e., triangles such that the side lengths and area are all integers ([BG]).

Im Zentrum dieses Artikels stehen Dreiecke, deren Seiten a, b, c eine arithmetische Progression bilden, d.h. $b = (a + c)/2$. Derartige Dreiecke sind weder besonders selten noch besonders häufig anzutreffen: In der Menge der Äquivalenzklassen ähnlicher Dreiecke bilden sie einen Raum der Kodimension eins, genau wie gleichschenklige oder rechtwinklige Dreiecke. Im Gegensatz zu den eben genannten Klassen scheinen aber Dreiecke, deren Seiten eine arithmetische Progression bilden, wenig untersucht worden zu sein, obwohl sie, wie die Autoren der vorliegenden Arbeit zeigen, hübsche geometrische Eigenschaften aufweisen, über die sie sich charakterisieren lassen.

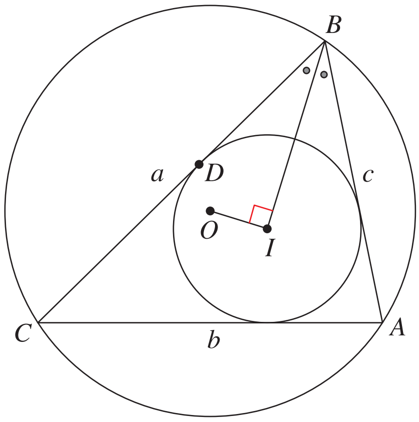

Fig. 1

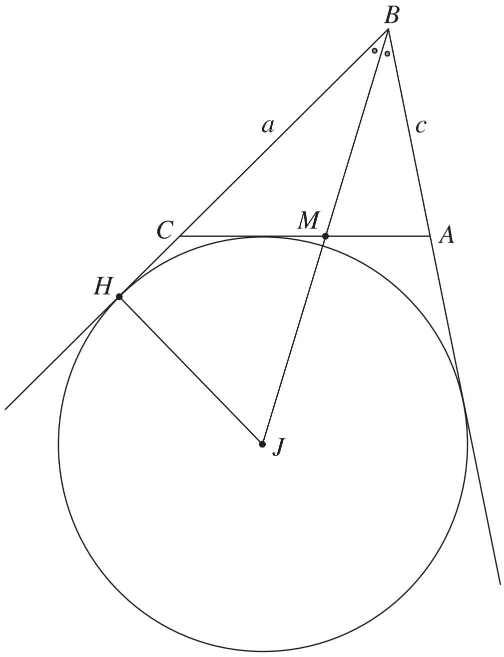

Fig. 2

(ii) *The line joining the incenter I and the circumcenter O is perpendicular to the line BI* (*see Figure* 1).

(iii) *The midpoint of B and the excenter opposite to B lies on the edge AC* (*see Figure* 2).

The condition (i) is given in an excercise in Section 8.6.7 on page 120 of a note by Paul Yiu which is available through the web ([Y]). The authors found the conditions (ii) and (iii) accidentally when they were drawing figures with softwares such as GeoGebra, Maple, and Mathematica. Later, they got informed that the condition (ii) has been given in Problem 1 of the Indian National Mathematical Olympiad 2006. Three proofs of (ii), using Stewart's theorem, Ptolemy's theorem, and Euler's theorem and the formula of $\cos(B/2)$ respectively, can be found in [I]. We give a (possibly) new proof of (ii), a little bit more elementary, but still using Euler's theorem. As for the condition (iii), the authors do not know references.

Let us first prepare formulae which are needed in our proof. Let r, R denote the radii of the incircle and the circumcircle respectively, S the area of $\triangle ABC$, and $s = (a+b+c)/2$. Then, Heron's formula states

$$S = \sqrt{s(s-a)(s-b)(s-c)}\,. \tag{1}$$

Since $|\triangle ABC| = |\triangle IAB| + |\triangle IBC| + |\triangle IAC|$ we have

$$r = \frac{2S}{a+b+c} = \frac{S}{s}. \tag{2}$$

The law of sines, $R = a/(\sin A)$, implies

$$R = \frac{abc}{4S}\,. \tag{3}$$

Finally, Euler's theorem (see, for example, [J, p. 186]) states

$$IO^2 = R^2 - 2rR. \tag{4}$$

Proof. (i) The line GI is parallel to the line AC if and only if the heights of G and I above the line AC are the same, which should be one third of the height of the vertex B. Therefore the condition (i) is equivalet to saying that $|\triangle IAC| = |\triangle ABC|/3$, which, by (2), is equivalent to $br = (a+b+c)r/3$.

(ii) Let D be the foot of the perpendicular to the edge BC from I (see Figure 1). Since D is the tangent point of the incircle with the side BC, we have $BD = (a-b+c)/2 = s-b$. Therefore,

$$\begin{aligned}
OI \perp BI &\iff BI^2 + IO^2 = OB^2 \\
&\overset{(4)}{\iff} \left(BD^2 + r^2\right) + (R^2 - 2rR) = R^2 \\
&\overset{(2)}{\iff} 2rR = (s-b)^2 + \frac{s(s-a)(s-b)(s-c)}{s^2} \\
&\overset{(2),(3)}{\iff} \frac{abc}{2s} = \frac{s-b}{s}\,(s(s-b) + (s-a)(s-c)) \\
&\iff abc = 2(s-b)(2s^2 - (a+b+c)s + ac) = 2(s-b)ac \\
&\iff b = a - b + c,
\end{aligned}$$

which completes the proof.

(iii) Let J be the excenter opposite to B, M the intersection of BJ and AC. We show that $2b = a + c$ if and only if $BJ = 2BM$.

Let us compute BJ first. Let H be the foot of perpendicular to the line BC from J (see Figure 1). Since H is the tangent point of the excircle and the line BC, we have $BH = (a+b+c)/2 = s$. On the other hand, since JH is the radius of the excircle, $JH = 2S/(a-b+c) = S/(s-b)$. Therefore,

$$\begin{aligned}
BJ^2 &= BH^2 + JH^2 \\
&\overset{(1)}{=} s^2 + \frac{s(s-a)(s-b)(s-c)}{(s-b)^2} \\
&= \frac{s}{(s-b)}\,(s(s-b) + (s-a)(s-c)) \\
&= ac\,\frac{a+b+c}{a-b+c}.
\end{aligned}$$

Next we compute BM. Let E be the intersection of the circumcircle of the triangle ABC and the line BM. Put $m = CM, n = AM, x = BM$, and $y = ME$. Since the triangle

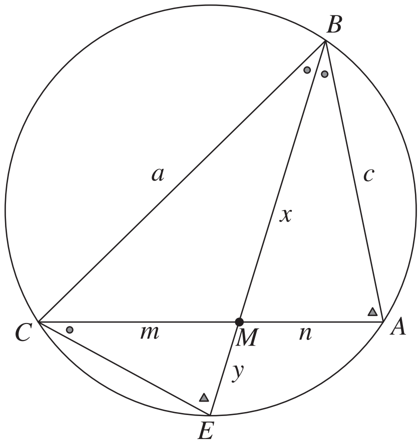

Fig. 3

BEC is similar to the triangle BAM (see Figure 3), we have $(x + y) : a = c : x$, i.e., $x^2 = ac - xy$. On the other hand, the secant theorem implies $xy = mn$. Since $m : n = a : c$,

$$x^2 = ac - mn = ac - \frac{ab}{a+c} \cdot \frac{cb}{a+c} = \frac{ac}{(a+c)^2}\left((a+c)^2 - b^2\right),$$

which implies

$$BM^2 = \frac{ac(a+b+c)(a-b+c)}{(a+c)^2}. \tag{5}$$

Therefore,

$$\begin{aligned} BJ = 2BM &\iff (a+c)^2 = 4(a-b+c)^2 \\ &\iff 2b = a + c, \end{aligned}$$

which completes the proof. □

Remark 2. We can also show the formula (5) by applying the law of cosines to $\triangle BAC$ and $\triangle BAM$.

Acknowledgement

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