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Parametric integrals, the Catalan numbers, and the beta function

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1 Introduction

Let a be a positive real number. For every non-negative integer n, Dana-Picard considered in [2] the definite integral

$$I_n(a) = \int_0^a x^n \sqrt{a^2 - x^2}\,\mathrm{d}\,x \tag{1.1}$$

and inductively obtained that

$$I_0(a) = \frac{a^2\pi}{4}, \quad I_1(a) = \frac{a^3}{3}, \quad \text{and} \quad I_n(a) = \frac{a^2(n-1)}{n+2} I_{n-2}(a)$$

In der mathematischen Physik spielen *Spezielle Funktionen* eine tragende Rolle. Zu den am besten untersuchten Exemplaren dieser Funktionen gehören die Gammafunktion und die Eulersche Betafunktion, die sich durch die Eulerschen Integrale zweiter respektive erster Gattung ausdrücken lassen. Der Autor der vorliegenden Arbeit untersucht Familien von parametrischen Integralen und den Wallis-Quotienten $W_n = \frac{(2n-1)!!}{(2n)!!}$ und drückt deren Werte mit Hilfe der Gamma- und der Betafunktion aus. Dabei ergibt sich eine überraschende Verbindung zu den Catalan-Zahlen: Diese treten in der Kombinatorik in zahlreichen und auf den ersten Blick ganz verschiedenen Abzählproblemen auf. Die gefundenen Formeln erlauben dem Autor inbesondere den Nachweis, dass die Folge der Catalan-Zahlen absolut konvex ist.

for $n \geq 2$. Furthermore, by telescopic methods in [1, 3], Dana-Picard derived in [2] that

$$I_{2p}(a) = \left(\frac{a}{2}\right)^{2p+2} \frac{(2p)!}{p!(p+1)!}\pi, \quad p \in \mathbb{N} \tag{1.2}$$

and

$$I_{2p+1}(a) = \frac{a^{2p+3}2^{2p+1}}{(2p+1)(2p+2)(2p+3)} \frac{p!(p+1)!}{(2p)!}, \quad p \geq 0. \tag{1.3}$$

Dana-Picard also observed in [2] that the quantities

$$C_p = \frac{(2p)!}{p!(p+1)!}, \quad p \geq 0$$

are just the Catalan numbers in combinatorics and that

$$C_p = \frac{1}{\pi} \int_0^2 x^{2p} \sqrt{4 - x^2}\, \mathrm{d}x, \quad p \in \mathbb{N}. \tag{1.4}$$

Dana-Picard further pointed out that the integral representation (1.4) is equivalent to

$$C_n = \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4-x}{x}}\, \mathrm{d}x, \quad n \geq 0, \tag{1.5}$$

which was obtained in [11] by the Mellin transform. For more information on the Catalan numbers C_n, please refer to the monographs [6, 25] and the paper [18] and plenty of literature cited therein.

In this paper, we will present a unified expression of the formulas (1.2) and (1.3) in terms of the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t, \quad \Re(z) > 0,$$

compute a new sequence of parametric integrals $\int_0^a \frac{x^n}{\sqrt{a^2-x^2}}\, \mathrm{d}x$ for $a > 0$ and $n \geq 0$ in terms of the gamma function Γ, discover the absolute convexity of the Catalan numbers C_n, compute a general sequence of parametric integrals

$$I(a; \alpha, \beta) = \int_0^a x^\alpha \left(a^2 - x^2\right)^\beta \mathrm{d}x \tag{1.6}$$

for $a > 0$ and $\alpha, \beta > -1$ in terms of the classical beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re(x), \Re(y) > 0,$$

and represent the above sequences of parametric integrals $\int_0^a x^n \left(a^2 - x^2\right)^{\pm 1/2} \mathrm{d}x$, the Catalan numbers C_n, and the Wallis ratio $W_n = \frac{(2n-1)!!}{(2n)!!}$ in terms of the classical beta function $B(x, y)$.

2 A unified expression of (1.2) and (1.3)

In this section, we present a unified expression of the formulas (1.2) and (1.3) in terms of the gamma function Γ as follows.

Theorem 2.1. *For $a > 0$ and $n \ge 0$, we have*

$$I_n(a) = a^{n+2} \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{4\Gamma\left(\frac{n}{2}+2\right)}. \tag{2.1}$$

Proof. By changing variables $x = a \sin s$ for $s \in \left[0, \frac{\pi}{2}\right]$, we have

$$\begin{aligned} I_n(a) &= a^{n+2} \int_0^{\pi/2} \sin^n s \sqrt{1-\sin^2 s}\, \cos s \,\mathrm{d}\, s \\ &= a^{n+2} \int_0^{\pi/2} \sin^n s \cos^2 s \,\mathrm{d}\, s \\ &= a^{n+2} \left[\int_0^{\pi/2} \sin^n s \,\mathrm{d}\, s - \int_0^{\pi/2} \sin^{n+2} s \,\mathrm{d}\, s \right]. \end{aligned}$$

Since

$$\int_0^{\pi/2} \sin^n s \,\mathrm{d}\, s = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, \quad n \in \mathbb{N},$$

see [12, Section 1.1.3], it follows that

$$I_n(a) = a^{n+2} \left[\frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} - \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+2}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}+1\right)} \right] = a^{n+2} \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}.$$

The proof of Theorem 2.1 is complete. ☐

3 A new sequence of parametric integrals

Differentiating on both sides of the equation (2.1) produces a new sequence of parametric integrals $\int_0^a \frac{x^n}{\sqrt{a^2-x^2}} \,\mathrm{d}\, x$ and a new integral representation for the Catalan numbers C_n.

Theorem 3.1. *For $a > 0$ and $n \ge 0$, we have*

$$\int_0^a \frac{x^n}{\sqrt{a^2-x^2}} \,\mathrm{d}\, x = \sqrt{\pi}\, a^n \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} \tag{3.1}$$

and, consequently,

$$C_n = \frac{2}{\pi(n+1)} \int_0^2 \frac{x^{2n}}{\sqrt{4-x^2}} \,\mathrm{d}\, x. \tag{3.2}$$

Proof. It is well known [16, Lemma 2.1] that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_0}^{t} f(x,t)\,\mathrm{d}x = f(t,t) + \int_{x_0}^{t}\frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x,$$

where $f(x,t)$ is differentiable in t and continuous in (x,t) in some region of the (x,t)-plane. Hence, differentiating with respect to a on both sides of (1.1) gives

$$I_n'(a) = a\int_0^a \frac{x^n}{\sqrt{a^2-x^2}}\,\mathrm{d}x. \tag{3.3}$$

On the other hand, differentiating with respect to a on both sides of (2.1) results in

$$I_n'(a) = \frac{\sqrt{\pi}}{4}(n+2)a^{n+1}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}. \tag{3.4}$$

Combining (3.3) with (3.4) and simplifying lead to the formula (3.1).

The formula (3.2) follows readily from combination of

$$C_n = \frac{4^n\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad n \ge 0 \tag{3.5}$$

in [6, p. 112, Eq. (5.5)] and (3.1). The proof of Theorem 3.1 is complete. □

4 Convexity of the Catalan numbers

It is common knowledge that a sequence $\{\mu_n\}_0^\infty$ is said to be convex if the inequality

$$\mu_{n+1} \le \frac{\mu_n + \mu_{n+2}}{2}$$

is valid for every $n \ge 0$. An infinitely differentiable function f on an interval I is called absolutely convex on I if $f^{(2k)}(x) \ge 0$ on I. See [9, p. 375, Definition 3], [13, p. 2731, Definition 4.5], [22, p. 617, Definiton 3], or [23, p. 3356, Definition 3]. A sequence $\{\mu_n\}_0^\infty$ is said to be absolutely convex if its elements are non-negative and its successive differences satisfy $\Delta^{2k}\mu_n \ge 0$ for $n, k \ge 0$, where

$$\Delta^k \mu_n = \sum_{m=0}^{k}(-1)^m\binom{k}{m}\mu_{n+k-m}.$$

It is clear that an absolutely convex function (sequence) must be convex.

Utilizing the integral representations (1.4), (1.5), and (3.2), we can derive the absolute convexity of the sequences C_n and $(n+1)C_n$.

Theorem 4.1. *The sequences C_n and $(n+1)C_n$ for $n \ge 0$ are both (absolutely) convex.*

Proof. The absolute convexity of the sequence C_n follows from the integral representations (1.4) or (1.5) and the absolute convexity of the functions $\int_0^2 x^t\sqrt{4-x^2}\,\mathrm{d}x$ and $\int_0^4 x^t\sqrt{\frac{4-x}{x}}\,\mathrm{d}x$ with respect to t.

The absolute convexity of the sequence $(n+1)C_n$ follows from rewriting the formula (3.2) as

$$(n+1)C_n = \frac{2}{\pi}\int_0^2 \frac{x^{2n}}{\sqrt{4-x^2}}\,\mathrm{d}x$$

and the absolute convexity of the function $\int_0^2 \frac{x^t}{\sqrt{4-x^2}}\,\mathrm{d}x$ with respect to t. The proof of Theorem 4.1 is complete. □

5 A general sequence of parametric integrals

Motivated by the proof of Theorem 2.1, we now compute a general sequence of parametric integrals (1.6) in terms of the beta function $B(x,y)$.

Theorem 5.1. *For $a>0$ and $\alpha,\beta>-1$, we have*

$$I(a;\alpha,\beta) = \int_0^a x^\alpha\left(a^2-x^2\right)^\beta\,\mathrm{d}x = \frac{1}{2}a^{\alpha+2\beta+1}B\left(\frac{\alpha+1}{2},\beta+1\right). \tag{5.1}$$

Proof. By changing variables $x=a\sin t$ for $t\in\left[0,\frac{\pi}{2}\right]$ as in the proof of Theorem 2.1, we have

$$\begin{aligned}
I(a;\alpha,\beta) &= \int_0^{\pi/2}(a\sin t)^\alpha\left[a^2-(a\sin t)^2\right]^\beta a\cos t\,\mathrm{d}t\\
&= a^{\alpha+2}\int_0^{\pi/2}\sin^\alpha t\left(1-\sin^2 t\right)^\beta\cos t\,\mathrm{d}t\\
&= a^{\alpha+2}\int_0^{\pi/2}\sin^\alpha t\cos^{2\beta+1} t\,\mathrm{d}t = \frac{1}{2}a^{\alpha+2\beta+1}B\left(\frac{\alpha+1}{2},\beta+1\right),
\end{aligned}$$

where we used in the last step the formula

$$\int_0^{\pi/2}\sin^{2a-1}\theta\cos^{2b-1}\theta\,\mathrm{d}\theta = \frac{1}{2}B(a,b),\quad \Re(a),\Re(b)>0 \tag{5.2}$$

in [10, p. 142, Eq. 5.12.2]. The proof of Theorem 5.1 is complete. □

6 Remarks

Making use of the formula (5.1), we now represent the sequences of parametric integrals $\int_0^a x^n\left(a^2-x^2\right)^{\pm 1/2}\mathrm{d}x$, the Catalan numbers C_n, and the Wallis ratio in terms of the classical beta function $B(x,y)$ in the form of remarks.

Remark 6.1. By the formula (5.1) in Theorem 5.1, the formula (2.1) can be rewritten in terms of the beta function $B(x, y)$ as

$$I_n(a) = \frac{1}{2}a^{n+2}B\left(\frac{n+1}{2}, \frac{3}{2}\right).$$

Remark 6.2. By virtue of (3.5) and Theorem 5.1, we immediately recover the relation

$$C_n = \frac{1}{\pi}I_{2n}(2) = \frac{1}{\pi}2^{2n+1}B\left(\frac{2n+1}{2}, \frac{3}{2}\right) \tag{6.1}$$

and the integral representation (1.4) for $n \ge 0$.

Remark 6.3. By Theorem 5.1, the formula (3.1) can be rewritten in terms of the beta function $B(x, y)$ as

$$\int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} \,\mathrm{d}\,x = \frac{1}{2}a^n B\left(\frac{n+1}{2}, \frac{1}{2}\right).$$

Remark 6.4. It was stated in [5] that

$$\int_0^{\pi/2} \sin^t x \,\mathrm{d}\,x = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{t+2}{2}\right)}, \quad t > -1. \tag{6.2}$$

See also [12, p. 16, Eq. (2.18)]. By (5.2), we can alternatively express the formula (6.2) in terms of the beta function $B(x, y)$ as

$$\int_0^{\pi/2} \sin^t x \,\mathrm{d}\,x = \frac{1}{2}B\left(\frac{t+1}{2}, \frac{1}{2}\right).$$

Remark 6.5. It is well known that the Wallis ratio is defined by

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(n + 1/2\right)}{\Gamma(n+1)}, \quad n \in \mathbb{N}.$$

This quantity has been studied and applied by many mathematicians. See [4, 17, 19], for example, and plenty of literature therein. The Wallis ratio can be expressed in terms of the beta function $B(x, y)$ as

$$W_n = \frac{1}{\pi}B\left(n + \frac{1}{2}, \frac{1}{2}\right), \quad n \in \mathbb{N}.$$

Remark 6.6. Since

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} \,\mathrm{d}\,t, \quad \Re(a), \Re(b) > 0,$$

see [10, p. 142, Eq. 5.12.1], the relation (6.1) can be rearranged as

$$\frac{C_n}{2^{2n}} = \frac{2}{\pi}\int_0^1 t^{n-1/2}(1-t)^{1/2} \,\mathrm{d}\,t.$$

This implies that the sequence $\frac{C_n}{2^{2n}}$ for $n \ge 0$ is completely monotonic. For more information on the Catalan numbers C_n, their generalizations, and their (completely monotonic) properties, please refer to the monographs [6, 25], the formerly published papers [7, 8, 14, 15, 18, 20, 21, 24] and plenty of references therein.

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