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Elemente der Mathematik

Short note On the sums of Fibonacci and Lucas sequences or the art of cancelling $1 - x$

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Let $\{F_n : n \in \mathbb{N}_0\} = \{0, 1, 1, 2, 3, 5, \ldots\}$, and $\{L_n : n \in \mathbb{N}_0\} = \{2, 1, 3, 4, 7, 11, \ldots\}$, be sequences of Fibonacci, and Lucas numbers. Their generating functions are

$$F(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n, \quad \text{and} \quad L(x) = \frac{2 - x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_n x^n.$$

Recently, Sury [3], and Marques [2] established the Fibonacci–Lucas relations

$$\sum_{k=0}^{n} 2^k L_k = 2^{n+1} F_{n+1}, \quad \text{and} \quad \sum_{k=0}^{n} 3^k L_k + \sum_{k=0}^{n+1} 3^{k-1} F_k = 3^{n+1} F_{n+1}.$$

The first identity was elegantly proved by Kwong [1].

These two identities can be generalized easily using generating functions.

Let $G(x) = g_0 + g_1 x + g_2 x^2 + \cdots + g_n x^n + \cdots$, be any generating function. It follows that the coefficient of x^n is

a) $m^n g_n$, for $G(mx)$, b) g_{n-1} for $xG(x)$, $n > 0$,

c) g_{n+1} for $\dfrac{G(x)}{x}$, $n \geqslant 0$, d) $\displaystyle\sum_{k=0}^{n} g_k$, for $\dfrac{G(x)}{1-x} = \displaystyle\sum_{j=0}^{\infty} g_j x^j \times \sum_{k=0}^{\infty} x^k$.

Then for $m > 0$, $\sum_{k=0}^{n} m^k L_k + (m-2) \sum_{k=0}^{n+1} m^{k-1} F_k$, is the coefficient of x^n in

$$\begin{aligned}
&\frac{L(mx)}{1-x} + \frac{(m-2)F(mx)}{mx(1-x)} \\
&\qquad = \frac{2 - mx}{(1-x)(1-(mx)-(mx)^2)} + \frac{(m-2)mx}{mx(1-x)(1-(mx)-(mx)^2)} \\
&\qquad = \frac{m}{1-(mx)-(mx)^2} = \frac{F(mx)}{x}.
\end{aligned}$$

So, for nonnegative integer n, and real $m > 0$, the identity

$$\sum_{k=0}^{n} m^k L_k + (m-2)\sum_{k=0}^{n+1} m^{k-1} F_k = m^{n+1} F_{n+1}$$

holds.

References

[1] H. Kwong, An alternate proof of Sury's Fibonacci–Lucas relation, *Amer. Math. Monthly*, 121 (2014), 514.

[2] D. Marques, A new Fibonacci-Lucas relation, *Amer. Math. Monthly*, 122 (2015), 683.

[3] B. Sury, A polynomial parent to a Fibonacci–Lucas relation, *Amer. Math. Monthly*, 121 (2014), 236.

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