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On some irrational decimal fractions, revisited

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1 Introduction

Proving the same theorem by using more approaches is necessary in the development of mathematical reasoning (see [7] and [8]). Problem-solving in different ways develops mathematical knowledge, and creativity in the students' mathematical way of thinking (see [7]).

In the secondary school – not only for higher level students – we usually prove that $\sqrt{2}$ and $\log_{10} 2$ are irrational. More general we can demonstrate $\sqrt[m]{N}$ is irrational, unless N is the mth power of an integer and $\log_n m$ is irrational if m and n are integers, one of which has a prime factor which the other lacks. In these cases we usually suppose that the given number is rational, therefore can be written in the form a/b, where a and b are integers, with $b > 0$, which after a few steps leads to a contradiction.

The irrationality of $\tan 1°$ can be easily proved as well with the help of the compound angle formulae for the tangent of the sum of two angles, using the fact that $\tan 30°$ is irrational.

Another way to prove the irrationality of a number is using the next theorem: if the real number x satisfies the equation $x^n + c_1 x^{n-1} + \cdots + c_n = 0$, with integral coefficients, then

Das hier behandelte Problem verknüpft Primzahlen mit irrationalen Zahlen: Wir bilden eine Dezimalzahl zwischen 0 und 1, indem wir nach dem Komma die Primzahlen als Ziffernfolge notieren, also 0.23571113 . . .. Diese Zahl ist irrational. Aber wie beweist man diese Tatsache möglichst elementar? Die Autoren der vorliegenden Arbeit sammeln bekannte Beweise, die auf dem Primzahlsatz von Dirichlet respektive auf Bertrands Postulat beruhen, und sie geben zwei neue, einfache Beweise.

x is either an integer or an irrational number. For example $\sqrt{2}+\sqrt{5}$ is irrational, since it is not an integer and satisfies the equation $x^4-14x^2+9=0$.

We have to mention that the irrationality of a number can be proved in a geometrical way as well (see [3]).

To prove that e is irrational is a bit more complicated and needs deeper mathematical knowledge, but if we define it as $\sum_{n=0}^{\infty}\frac{1}{n!}$, then this can be proved easily in different ways, one of which assumes that the theorem is false, and then deduces that a certain number is integer, positive and less than one, obviously a contradiction. For another simple proof for the irrationality of e see [4]. To prove that π is irrational is much more complicated than all those mentioned above, the required mathematical knowledge exceeds the secondary school curriculum.

Another interesting question is whether a^b can be rational for both a and b irrational. Let $a=\sqrt{2}^{\sqrt{2}}$ and $b=\sqrt{2}$. Then either a is rational (and an example for the question), or a is irrational and $a^b=2$ an example. We mention that e^{π} and $2^{\sqrt{2}}$ are irrational numbers (proven), but we still just suspect that 2^e, π^e, and $\pi^{\sqrt{2}}$ are also.

Before we turn to the main aim of this article we show a problem that can serve as a prelude of it: if we take $\gamma = 0.149162536\ldots$ where the decimal point is followed by the consecutive positive square numbers, than γ is irrational. Namely in the decimal representation of γ there are series of zeros of any length (10^{2k} is a square number if k is a natural number). The numbers that can be expressed as recurring decimals are precisely the rational numbers. But it is not possible, that the repeating string in the decimal representation of γ contains only non-zero digits.

A frequently mentioned result in elementary number theory is the following theorem:

Theorem 1. *Let $\mathcal{P}=\{2=p_1<p_2<\cdots<p_n<\ldots\}$ be the infinite sequence of prime numbers. Denote by $\langle p_n\rangle$ the decimal expansion of its elements. Then the infinite decimal fraction*

$$\alpha = 0.\langle p_1\rangle\langle p_2\rangle\ldots\langle p_n\rangle\ldots$$

is an irrational number.

The main goal of our paper is to give a new and simple proof of Theorem 1. We mention here that our arguments go through for an arbitrary base; nevertheless in the literature the decimal case is the well-known, so we concentrate on this in the rest.

The two usual different proofs are based on the following deep theorems respectively:

Theorem 2 (Dirichlet)**.** *For any two positive coprime integers a, b, the sequence $\{an+b : n=1,2,\ldots\}$ contains infinite many prime numbers.*

Theorem 3 (Bertrand's postulate)**.** *For every integer $n>3$ the interval $[n,2n]$ contains at least one prime number.*

The proofs of Theorem 2 and 3 are complicated and rather long. In Section 3 we recall a proof which appeared in The American Mathematical Monthly (see [5]) and in Section 4 we show a new and simpler way to show Theorem 1.

Definition 1. We say that a decimal fraction $\beta = 0.a_1a_2\dots a_n\dots$ $(a_n \in \{0, 1, \dots 9\})$ contains the pattern $b_1b_2\dots b_k$ $(b_m \in \{0, 1, \dots 9\};\ m = 1, 2, \dots k)$ if there is a subscript i such that $a_i = b_1, a_{i+1} = b_2, \dots, a_{i+k-1} = b_k$.

2 The usual two proofs of Theorem 1

Proof. Theorem 2 ⇒ Theorem 1. The examined α number is not a terminating decimal (since the number of prime numbers is infinite), with infinite digits different from 0. Let us suppose that α is a rational number. Then $\alpha = 0.a_1a_2\dots a_k\dot{b}_1\dots\dot{b}_n\dot{b}_1\dots\dot{b}_n\dots$, where a_i and b_i are digits and $\dot{b}_1\dots\dot{b}_n$ is the repeating pattern. In the repeating string not all digits are 0, so among any n consecutive digits after a_k there is at least one which is not 0. From Theorem 2 with $a = 10^{n+1}$ and $b = 1$ we get, that there are infinite prime numbers with 1 as the last digit, preceded by n consecutive 0. Since this is repeated infinitely often, in α there are infinitely often n consecutive 0 digits, which leads to a contradiction. □

Proof. Theorem 3 ⇒ Theorem 1. Again assume that α is a rational number

$$\alpha = 0.a_1a_2\dots a_k\dot{b}_1\dots\dot{b}_n\dot{b}_1\dots\dot{b}_n\dots,$$

where a_i and b_i are digits and $\dot{b}_1\dots\dot{b}_n$ is the repeating pattern.

From Theorem 3 we get that for any m natural number $(m > 0)$ there are at least two prime numbers with m digits. Indeed each interval $[10^m, 2\cdot 10^m]$ and $[2\cdot 10^m, 4\cdot 10^m]$ contain different prime numbers. Let m be a multiple of n, so that the two prime numbers with m digits make part of the repeating string. Since α is a repeating decimal all digits of one of the prime numbers recur. So the two prime numbers are equal, which is a contradiction. □

3 First simpler approach

The second named author observed (see [5]) that there is a simpler proof of this statement which uses just the fact that the series $\sum_{i_1}^{\infty}\frac{1}{p_i}$ diverges (one can find this proof in [6] and [9] too).

For the sake of completeness, let us briefly repeat here this argument: if the decimal fraction would contain a periodical pattern $B := b_1b_2\dots b_k$ $(b_m \in \{0, 1, \dots 9\};\ m = 1, 2, \dots k)$ then let $C := c_1c_2\dots c_{2k}$ $(c_m \in \{0, 1, \dots 9\};\ m = 1, 2, \dots 2k)$ be a pattern with length $2k$ which does not contain pattern B.

Let $S_C := \{n \in \mathbb{N} :\ n \text{ does not contain the pattern } C\}$.

Lemma 1. *The series*

$$\sum_{n\in S_C}^{\infty}\frac{1}{n}$$

converges.

It implies that there are infinitely many prime numbers containing the pattern C $\left(\sum_{i_1}^{\infty}\frac{1}{p_i}\right.$ diverges$\left.\right)$ which avoids the periodical pattern B. It leads to a contradiction. □

In the next section we give a new simple proof. For this proof we need a stronger form of the previous lemma:

Lemma 2. *Let C be an arbitrary pattern with length m. Then there are two positive real numbers $A > 0$ and $0 < B < 1$ such that*

$$|S_C \cap \{1, 2, \ldots, x\}| < Ax^B.$$

One can conclude by undergraduate calculus that Lemma 2 implies Lemma 1.

Proof of Lemma 2. Instead of base 10, express the positive integers in base $q := 10^m$. Let

$$S_q := \{n \in \mathbb{N} : \ n \text{ does not contain the digit } \langle c_1 c_2 \ldots c_m \rangle \text{ in base } q\}.$$

Clearly $S_C \subset S_q$. Thus it is enough to show that $|S_q \cap \{1, 2, \ldots, x\}| < Ax^B$.
Write $S_k := S_0 \cap [q^k, q^{k+1})$; i.e., we collect to S_k all integers from S_q having k many digits. Let $n \in S_k$, $n = x_1 q^{k-1} + x_2 q^{k-2} + \cdots + x_k$. x_1 differs from $\langle 0, 0 \ldots 0 \rangle$ and from $\langle c_1 c_2 \ldots c_m \rangle$. The other digits differ from $\langle c_1 c_2 \ldots c_m \rangle$. Hence $|S_k| = (q-2)(q-1)^{k-1}$. Define T by $q^T \le x < q^{T+1}$. Thus

$$\begin{aligned} |S_q \cap \{1, 2, \ldots, x\}| &= \sum_{k=1}^{T} (q-2)(q-1)^{k-1} = (q-2)\frac{(q-1)^T - 1}{q-2} \\ &< (q-1)^T = q^{T \ln(q-1)/\ln q} < x^{\ln(q-1)/\ln q} = x^B, \end{aligned}$$

where $0 < B = \ln(q-1)/\ln q < 1$. □

4 Second simpler approach

Our strategy in the second simpler proof will be the following:

Fact 1. *Let $C := c_1 c_2 \ldots c_{2k}$, $c_m \in \{0, 1, \ldots 9\}$; $m = 1, 2, \ldots 2k$ be an arbitrary pattern in base* 10.
There are infinitely many prime numbers containing pattern C in base 10.

Firstly we prove

Theorem 4. *We have*

$$\pi(x) > \frac{\ln 2}{2} \frac{x}{\ln x},$$

$\big($*where* $\frac{\ln 2}{2} = 0.3465\ldots\big)$

Proof. We split our proof into three parts.

1. Let

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

be a non-zero polynomial with integer coefficients for which $p(x) \ge 0$ for every $x \in [0, 1]$. Since $p(x)$ is a continuous function, $p(x) \not\equiv 0$ we get by the Leibnitz–Newton rule

$$0 < \int_0^1 p(x)dx = \left[a_n \frac{x^{n+1}}{n+1} + \cdots + a_1 \frac{x^2}{2} + a_0 x \right]_0^1 = \frac{A}{l.c.m.[1, 2, \ldots, n+1]},$$

where A is a positive integer. Thus $A \geq 1$ and so

$$\int_0^1 p(x)dx \geq \frac{1}{l.c.m.[1,2,\ldots,n+1]}.$$

2. How can be calculated $l.c.m[1,2,\ldots,N]$, what is the prime factorization of it?

Example 1. $l.c.m[1,2,\ldots,10]$: here the prime factors are $2,3,5,7$. The power of 2 is three, the prime factor 3 has power two, 5 and 7 have power one.
Thus $l.c.m.[1,2,\ldots,10] = 2^3 \cdot 3^2 \cdot 5 \cdot 7$.

Now we turn to the general calculation: assume that in the factorization of

$$l.c.m.[1,2,\ldots,N]$$

the occurring primes are $p_1, p_2, \ldots, p_k$. Clearly $k = \pi(N)$ and we have

$$p_i^{\alpha_i} \leq N < p_i^{\alpha_i+1}.$$

Hence

$$l.c.m.[1,2,\ldots,N] = \prod_{i=1}^{k} p_i^{\alpha_i} \leq N^k = N^{\pi(N)}.$$

Thus by 1

$$\int_0^1 p(x)dx \geq \frac{1}{l.c.m.[1,2,\ldots,n+1]} \geq \frac{1}{(n+1)^{\pi(n+1)}}.$$

3. Finally we give an explicit polynomial $p(x)$. Let $p(x) := (x(1-x))^k$. Its coefficients are integers and clearly for every $x \in [0,1]$ $x(1-x)$ is a parabola cupped down. The numbers 0 and 1 are roots, $p(x)$ is non-negative in $(0,1)$, not identically zero, hence the function $(x(1-x))^k$ so does. Here $x(1-x) \leq 1/4$ thus

$$p(x) = (x(1-x))^k \leq \frac{1}{4^k} \quad \text{for all } x \in [0,1].$$

Hence

$$\int_0^1 p(x)dx = \int_0^1 (x(1-x))^k\,dx \leq \int_0^1 \frac{1}{4^k}dx = \frac{1}{4^k}.$$

The degree of $p(x)$ is $n = 2k$. Thus by 1. and 2.

$$\frac{1}{(n+1)^{\pi(n+1)}} = \frac{1}{(2k+1)^{\pi(2k+1)}} \leq \int_0^1 p(x)dx \leq \frac{1}{4^k}.$$

Rearranging

$$4^k \leq (2k+1)^{\pi(2k+1)}$$

and taking the logarithm

$$2k\ln 2 \leq \pi(2k+1)\ln(2k+1), \quad \text{i.e.,} \quad \ln 2\frac{2k}{\ln(2k+1)} \leq \pi(2k+1).$$

Since $2k > \frac{2k+1}{2}$, $\pi(2k+1) = \pi(2k)$ for $k > 1$ and $\ln(2k+1) < 2\ln(2k)$ we have both

$$\frac{\ln 2}{2}\frac{2k}{\ln(2k)} \leq \pi(2k) \quad \text{and} \quad \frac{\ln 2}{2}\frac{2k+1}{\ln(2k+1)} \leq \pi(2k+1),$$

i.e., shortly

$$\pi(x) > \frac{\ln 2}{2}\frac{x}{\ln x}.$$

Remarks. This argument is due to Gelfond (see, e.g., in [2]).

Now we are in the position to complete our proof: we have to show

Lemma 3. *Let* $0 < B < 1$. *We have*

$$\lim_{x\to\infty} \frac{x/\ln x}{x^B} = \infty.$$

Proof of Lemma 3. By the L'Hospital rule

$$\lim_{x\to\infty} \frac{x/\ln x}{x^B} = \lim_{x\to\infty} \frac{x^{1-B}}{\ln x} = \lim_{x\to\infty} \frac{(1-B)x^{-B}}{1/x} = \lim_{x\to\infty} (1-B)x^{1-B} = \infty,$$

i.e., $\mathcal{P} \setminus S_C$ is a non-empty set. □

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