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A short combinatorial proof of derangement identity

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1 Introduction

Having a permutation $\sigma \in S_n$, $\sigma : [n] \to [n]$ where $[n] := \{1, 2, \ldots, n\}$, it is said that $k \in [n]$ is a *fixed point* if it is mapped to itself, $\sigma(k) = k$. Permutations without fixed points are of particular interest and are usually called *derangements*. We let D_n denote the number of derangements of the set $[n]$, $D_n = |S_n^{(0)}|$,

$$S_n^{(0)} := \{\sigma \in S_n : \sigma(k) \neq k, k = 1, \ldots, n\}.$$

Derangements are usually introduced in the context of the inclusion-exclusion principle

Die Subfakultät $!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \left\lfloor \frac{n!+1}{e} \right\rfloor$ gibt die Anzahl der Derangements, d.h. der fixpunktfreien Permutationen von n unterscheidbaren Objekten an. Die Anzahl der Permutationen dieser Objekte mit genau r Fixpunkten wird als Rencontres-Zahl bezeichnet. Es gibt eine Vielzahl interessanter kombinatorischer Identitäten, bei denen die Subfakultät und Rencontres-Zahlen eine Rolle spielen. Am bekanntesten ist vermutlich das Inklusions-Exklusionsprinzip. In der vorliegenden Arbeit betrachten die Autoren gewichtete Summen von Subfakultäten. Diese lassen sich zwar auch algebraisch beweisen, hier werden jedoch elegante kombinatorische Abzählargumente für die Herleitung benützt.

[3, 5, 10], since this principle is used to provide an interpretation of D_n as a *subfactorial*,

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{1}$$

The numbers $D_0, D_1, D_2, \ldots, D_n, \ldots$ form the recursive sequence $(D_n)_{n \geq 0}$ defined by the recurrence formula

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \tag{2}$$

and initial terms $D_0 = 1$, $D_1 = 0$ [7]. There is a counting argument to prove this. Let the number k be mapped by σ to the number j, $j = 1, \ldots, k-1, k+1, \ldots, n$. Note that there are $(n-1)$ such permutations σ. Now, we separate the set of permutations σ into two disjoint sets $\mathcal{A}$ and $\mathcal{B}$, such that

$$\mathcal{A} := \{\sigma \in S_n^{(0)} : \sigma(j) \neq k, \sigma(k) = j\}$$
$$\mathcal{B} := \{\sigma \in S_n^{(0)} : \sigma(j) = k, \sigma(k) = j\}.$$

This means that

$$D_n = (n-1)(|\mathcal{A}| + |\mathcal{B}|).$$

The set $\mathcal{A}$ counts D_{n-1} elements while the set $\mathcal{B}$ counts D_{n-2} elements. The fact that the number k in this reasoning is chosen without loss of generality, completes the proof of (2).

There is another recurrence for the sequence $(D_n)_{n \geq 0}$,

$$D_n = nD_{n-1} + (-1)^n. \tag{3}$$

Namely, set $\delta_n = D_n - nD_{n-1}$ for every $n \geq 1$. Then $\delta_1 = -1$, and formula (2) implies that one has for every $n \geq 2$

$$\delta_n = D_n - nD_{n-1} = (n-1)D_{n-2} - D_{n-1} = -\delta_{n-1},$$

hence one gets immediately $\delta_n = (-1)^n$, which proves (3).

When we iteratively apply recurrence (3) to the derangement number on the r.h.s. of this relation we get

$$nD_{n-1} + (-1)^n = n\big[(n-1)D_{n-2} + (-1)^{n-1}\big] + (-1)^n$$

which finally results with

$$n(n-1)(n-2)\cdots 3(-1)^2 + n(n-1)(n-2)\cdots 4(-1)^3 + \cdots + (-1)^n. \tag{4}$$

on the r.h.s. of (3), which completes the proof of (1).

A few identities for the sequence $(D_n)_{n \geq 0}$ are known [4, 6, 8]. In [4] Deutsche and Elizalde give a nice identity

$$D_n = \sum_{k=2}^{n} (k-1) \binom{n}{k} D_{n-k}. \tag{5}$$

Recently, Bhatnagar presents families of identities for some sequences including the shifted derangement numbers [1], deriving it using an Euler identity [2]. In what follows we demonstrate a combinatorial proof for that derangement identity, with weighted sum.

2 A pair of weighted sums for derangements

We define the *rencontres number* $D_n(r)$ as the number of permutations $\sigma \in S_n$ having exactly r fixed points. Thus, $D_n(0) = D_n$. For a given $r \in \mathbb{N}$, we define the sequence $D_0(r), D_1(r), \dots, D_n(r), \dots$, denoted by $(D_n(r))_{n\geq r}$.

Applying an analogue counting argument that we used when proving relation (2), one can represent rencontres numbers by the derangement numbers,

$$D_n(r) = \binom{n}{r} D_{n-r}. \tag{6}$$

On the other hand, relation (6) follows immediately from the fact that fixed points here are r-combinations over the set of n elements.

A few other notable properties of the rencontres numbers are also known. It follows from (3) that $D_n - D_n(1) = (-1)^n$ for every $n \geq 1$. According to the definition of rencontres numbers, the sum of the nth row in the array of numbers $(D_n(r))_{n\geq r}$ is equal to $n!$,

$$n! = \sum_{k=0}^{n} D_n(k). \tag{7}$$

Moreover, identity (6) shows that D_n can be interpreted as a weighted sum of rencontres numbers in the nth row of the array, by means of relation (5),

$$D_n = \sum_{k=2}^{n} (k-1) D_n(k). \tag{8}$$

The number $D_n/(n-1)$ is also a weighted sum of previous consecutive derangement numbers. For example, $24 + 12D_2 + 4D_3 + D_4 = \frac{D_6}{5}$. In general we have

$$n! + \sum_{k=2}^{n} \frac{n!}{k!} D_k = \frac{D_{n+2}}{n+1}, \tag{9}$$

as follows from Theorem 1.

Theorem 1. *For $n \in \mathbb{N}$ and the sequence of derangement numbers $(D_n)_{n\geq 0}$ we have*

$$1 + \sum_{k=1}^{n} \frac{D_k}{k!} = \frac{D_{n+2}}{(n+1)!}. \tag{10}$$

Proof. Within a derangement σ, the number k, $k = 1, \dots, n$ can be mapped to any j, $j = 1, \dots, k-1, k+1, \dots, n$. We let $\mathcal{A}_n$ denote the set of derangements with $\sigma(k) = j$, where $j \neq k$,

$$\mathcal{A}_n := \{\sigma \in S_n^{(0)} : \sigma(k) = j\}.$$

Obviously, cardinality of the set $\mathcal{A}_n$ is independent of j, $j \neq k$. More precisely,

$$|\mathcal{A}_n| = \frac{D_n}{n-1}.$$

Furthermore, we separate the set $\mathcal{A}_n$ into two disjoint sets of derangements, sets $\mathcal{B}_n$ and $\mathcal{C}_n$,

$$\mathcal{B}_n := \{\sigma \in \mathcal{A}_n : \sigma(j) = k\}$$
$$\mathcal{C}_n := \{\sigma \in \mathcal{A}_n : \sigma(j) \neq k\}.$$

Obviously, the set $\mathcal{B}_n$ counts D_{n-2} elements. For derangements in $\mathcal{C}_n$ there are now $(n-2)$ equivalent ways to map j (excluding j and k), as Figure 1 illustrates. Thus, we have

$$|\mathcal{C}_n| = (n-2)|\mathcal{A}_{n-1}|,$$

which gives the recurrence relation

$$|\mathcal{A}_n| = D_{n-2} + (n-2)|\mathcal{A}_{n-1}|. \tag{11}$$

After repeating usage of (11) we get identity (9) which completes the proof. □

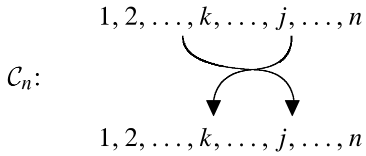

Figure 1 In case of derangements in the set $\mathcal{C}_n$ there are $(n-2)$ equivalent ways to map j.

In order to prove Theorem 1 algebraically, we apply recurrence (2) to get

$$\begin{aligned}\frac{D_{n+2}}{(n+1)!} &= \frac{(n+1)(D_{n+1}+D_n)}{(n+1)!} = \frac{D_{n+1}}{n!} + \frac{D_n}{n!}\\ &= \frac{n(D_n + D_{n-1})}{n!} + \frac{D_n}{n!} = \frac{D_n}{(n-1)!} + \frac{D_{n-1}}{(n-1)!} + \frac{D_n}{n!}\\ &= 1 + \frac{D_1}{1!} + \cdots + \frac{D_n}{n!} = 1 + \sum_{k=1}^{n} \frac{D_k}{k!}.\end{aligned}$$

Theorem 2. *For $n \in \mathbb{N}$ and the sequence of derangement numbers $(D_n)_{n\geq 0}$ we have*

$$1 + \sum_{k=1}^{n} \frac{(-1)^k D_{k+3}}{k+2} = (-1)^n D_{n+2}. \tag{12}$$

Proof. By applying recurrence (2) we have

$$\begin{aligned}\sum_{k=0}^{n} \frac{(-1)^k D_{k+3}}{k+2} &= \frac{2(D_2+D_1)}{2} - \frac{3(D_3+D_2)}{3} + \cdots + (-1)^n \frac{(n+2)(D_{n+2}+D_{n+1})}{n+2}\\ &= (D_2 + D_1) - (D_3 + D_2) + \cdots + (-1)^n (D_{n+2} + D_{n+1})\\ &= (-1)^n D_{n+2}\end{aligned}$$

which completes the proof. □

Once having Theorem 1, substitution of (6) in identity (10) gives the generalization (13).

$$1+\sum_{k=1}^{n}\frac{D_{k+r}(r)}{k!\binom{k+r}{r}}=\frac{D_{n+r+2}(r)}{(n+1)!\binom{n+r+2}{r}}. \tag{13}$$

The identity (14) follows by substitution of (6) in (12),

$$1+\sum_{k=1}^{n}\frac{(-1)^k D_{k+r+3}(r)}{(k+2)\binom{k+r+3}{r}}=\frac{(-1)^n D_{n+r+2}(r)}{\binom{n+r+2}{r}}. \tag{14}$$

Note that the terms in identity (14) are always integers, which can be seen as a consequence of recurrence relation (2).

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