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Short note An algorithm for nontransitive partitions

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Abstract. One of the most famous nontransitive games is given by the Efron dice, invented by the American statistician Bradley Efron. It consists of a set of four dice A, B, C, D which are labeled such that, in the long run, A beats B, B beats C, C beats D, and D beats A. In this paper, we present a new algorithm for a partition of the first $(2n+1)^2$ natural numbers in equally spaced parts such that the sums of the parts are equal and such that a nontransitive loop of equal probabilities arises. By this we find a method for the construction of what we call Efron partitions. Finally a labeling for five nontransitive icosahedrons is given.

1 Introduction

The Efron dice are well known as an example of the nontransitivity of probabilities [1]. In order to get a set of only three nontransitive dice we number the faces of three Laplace dice: A by 1, 1, 6, 6, 8, 8, B by 3, 3, 5, 5, 7, 7 and the faces of die C by 2, 2, 4, 4, 9, 9. A quick calculation shows: $P(A \to B) = P(B \to C) = P(C \to A) = \frac{5}{9}$, where $P(A \to B)$ stands for the probability that die A wins against die B. Furthermore the sums of the numbers on each die are equal. Is there a suchlike nontransitive labeling for icosahedrons, or more generally for n-faced dice?

H. Steinhaus and S. Trybula [2], as well as R.P. Savage [3], showed how to get a nontransitive loop of probabilities for three n-faced dice, when the sums of the numbers on each die do not have to be equal.

2 An algorithm for nontransitive partition

Definition 1. A basic nontransitive partition of the set $S_n = \{1, 2, \ldots, n^2\}$, $n \in \mathbb{N}$, is a equally spaced partition $\{X_i\}$, $i = 1, \ldots, n$, in which every part has the same sum together with a Laplace probability P with a nontransitivity condition:

1. $|X_i| = |X_j|,\ i, j = 1, \ldots, n$
2. $\sum_{a \in X_i} a = \sum_{a \in X_j} a$
3. $P(X_{i+1} \to X_i) = P(X_1 \to X_n) > \frac{1}{2},\ i = 1, \ldots, n-1,$

   $P(X_i \to X_j) = \frac{|\{x_k | x_k > x_m, x_k \in X_i, x_m \in X_j\}|}{(n)^2}, i, j = 1, \ldots, n.$

Theorem 2. *The algorithm below produces a basic nontransitive partition of the set* S_{2n+1}*,* $n \in \mathbb{N}$*, with* $P(X_{i+1} \rightarrow X_i) = P(X_1 \rightarrow X_{2n+1}) = \frac{2n^2+3n}{(2n+1)^2}$*,* $i = 1, \ldots, 2n$*.*

Algorithm. We explain the procedure of the algorithm by partitioning S_7. We start on the left-hand bottom of a 7×7 grid filling in upwards the natural numbers in the first column. If this is done we go to the element one row below and in the column just one step to the right. Again we fill upwards the second column, thinking the horizontal edges of the grid glued together. And so on until the grid is filled up.

X_7	7 ↘	9	18	27	29 ↑	38	47
X_6	6	8 ↑	17	26	35 ↘	37	46
X_5	5	14 ↘	16	25	34	36 ↑	45
X_4	4	13	15 ↑	24	33	42 ↘	44
X_3	3	12	21 ↘	23	32	41	43 ↑
X_2	2	11	20	22 ↑	31	40	49
X_1	1↑	10	19	28 ↘	30	39	48
	C_1	C_2	C_3	C_4	C_5	C_6	C_7

Due to the filling procedure each number in column C_j is greater than each number in column C_i, for $j > i$. In comparing row X_{i+1} to row X_i, $i = 1, \ldots, 2n$, and X_1 to X_{2n+1} we see that there are $2n$ numbers bigger by 1 than the number just below, but where the smallest and the biggest number in a column meet the difference is $-2n$, so the sum in each row must be equal. By the arguments above X_{i+1} wins against X_i, $i = 1, \ldots, 2n$, and X_1 wins against X_{2n+1}, in $(2n+1)+2n+(2n-1)+\cdots+2+1-1 = 2n^2+3n$ cases out of $(2n+1)^2$. The minus one in the forthcoming addition corrects the count where the smallest and the biggest number in a column meet. So $P(X_{i+1} \rightarrow X_i) = P(X_1 \rightarrow X_{2n+1}) = \frac{2n^2+3n}{(2n+1)^2}$, $i = 1, \ldots, 2n$.

3 A nontransitive partition for five icosahedrons and some Efron partitions

Let $L = [\ldots, \ldots, \ldots](k)$ be a list in which every entry appears exactly k-times.

Definition 3. A nontransitive partition of a list $\mathcal{L}_n = [1, 2, \ldots, n^2](k)$, $n, k \in \mathbb{N}$, is a equally spaced partition $\{L_i\}$, $i = 1, \ldots, n$, in which every part has the same sum together with a Laplace probability P with a nontransitivity condition such that 1., 2. and 3. of Definition 1 hold.

We call such a list $\mathcal{L}_n$ together with a nontransitive partition $\{L_i\}$, $i = 1, \ldots, n$, a (n^2, k)-Efron partition.

The nontransitive partitions of the sets S_{2n+1} in Theorem 2.2 are all $((2n+1)^2, 1)$-Efron partitions. The well-known example in the introduction is a $(3^2, 2)$-Efron partition.

Example. An icosahedron has $20 = 5 \cdot 4$ faces. We produce with our algorithm a basic nontransitive partition of the set S_5. Then we simply quadruple each number in every part to get a nontransitive partition with 100 numbers for the faces of 5 icosahedrons. This results in the partition:

$$\begin{aligned} L_1 &= [1, 8, 15, 17, 24](4), \\ L_2 &= [2, 9, 11, 18, 25](4), \\ L_3 &= [3, 10, 12, 19, 21](4), \\ L_4 &= [4, 6, 13, 20, 22](4), \\ L_5 &= [5, 7, 14, 16, 23](4). \end{aligned}$$

With $P(L_{i+1} \to L_i) = P(L_1 \to L_5) = \frac{14}{25}, i = 1 \ldots 4$.

We have just constructed a $((2n+1)^2, 4)$-Efron partition for $n = 2$. By an analoguous argument we get more generally:

Theorem 4. *If* $m = (2n+1)^2 \cdot k$, $n, k \in \mathbb{N}$, *then we can construct a* $((2n+1)^2, k)$*-Efron partition with* $P(L_{i+1} \to L_i) = P(L_1 \to L_{2n+1}) = \frac{2n^2+3n}{(2n+1)^2}$, $i = 1, \ldots, 2n$.

References

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