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Distances in rectangles and parallelograms

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Problem 11057 of the Am. Math. Monthly. This meanwhile quite popular problem (see [8] and [3]) is the following:

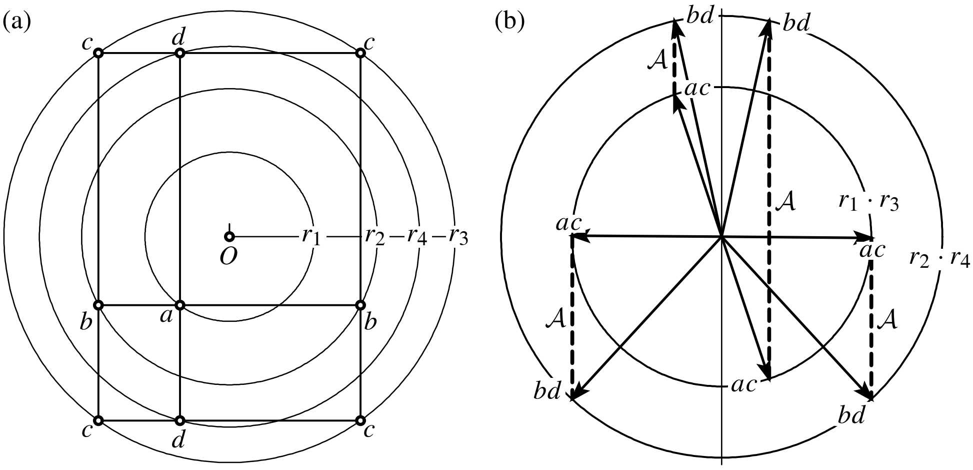
Find the largest (*and smallest*) *rectangle with vertices* a, b, c, d *on four prescribed concentric circles of radii* r_1, r_2, r_3, r_4 *respectively* (see Figure (a)).

This problem is originally due to the German mathematician Raphael Muhr [9]. Muhr's (unpublished) solution uses trigonometric functions and Lagrange multipliers. Other (published) solutions, in particular the one in [3], are up to 8 pages long. The solution in [8] uses the Cauchy–Schwarz inequality and, in [1], differential calculus, both followed by calculations. The short solution given here shows the result geometrically.

Solution. For an arbitrary point a, chosen on the innermost circle of radius r_1, there are, parallel to the axes, two pairs of points b and d on the median circles of radius r_2 and r_4 respectively. The four points c which complete the rectangles then lie all on the circle of radius r_3 for which

$$r_1^2 + r_3^2 = r_2^2 + r_4^2 \tag{1}$$

Manche auf den ersten Blick elementaren Probleme erweisen sich als Knacknüsse und kreisen eine ganze Weile in der mathematischen Community. Die ersten Beweise sind oft unbefriedigend und hinterlassen das Gefühl, noch nicht die wahre Natur des Problems erkannt zu haben. Die Autoren dieses Artikels haben es geschafft, für zwei klassische Probleme der Elementargeometrie mit Hilfe einer einfachen Identität in den komplexen Zahlen Beweise zu formulieren, die einen nicht nur von der Richtigkeit der Lösung überzeugen, sondern auch die erhellende Einsicht bringen.



must hold. This follows either from several applications of Pythagoras or more directly from Pappus VII.122 (also called "Apollonius-Pappus" or "the parallelogram identity" or "the median theorem") applied to the triangles bOd and aOc, which have both the same midpoint m of the diagonals bd and ac (see Figure (a)). Equation (1) was also the object of "Problem 11558" in the Am. Math. Monthly (see [5] and [4]).

We now consider a, b, c, d as points in the complex plane. Since they form a rectangle, we have

$$a + c = b + d \tag{2}$$

($=$ twice the coordinates of the mid-point m). Under this condition, a simple calculation shows the algebraic identity

$$(a - z)(c - z) - (b - z)(d - z) = ac - bd\,, \tag{3}$$

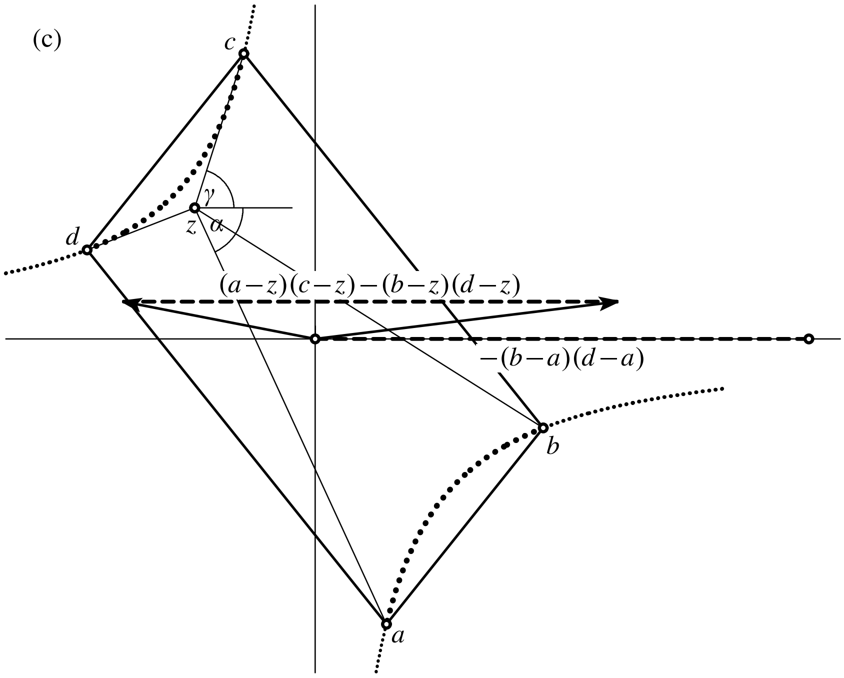
which is independent of z. If we set $z = a$ this gives

$$-(b - a)(d - a) = ac - bd\,. \tag{4}$$

Since by our choice of directions, $b-a$ is real and $d-a$ is purely imaginary, the expression $ac - bd$ must be purely imaginary, i.e., ac and bd must have the same real part (see Figure (b)). Thereby, ac and bd rotate on circles of radii $r_1 \cdot r_3$ and $r_2 \cdot r_4$ respectively. The difference of their imaginary parts is $|(b - a)(d - a)|$, i.e. the area $\mathcal{A}$ of the rectangle. It is thus clear, that the maximal and minimal area are reached when both ac and bd are purely imaginary, and are thus equal to $r_1 \cdot r_3 + r_2 \cdot r_4$ and $|r_1 \cdot r_3 - r_2 \cdot r_4|$.

We see that the upper right-most and lower left-most rectangles in the left picture are close to optimality.

The Goldsheid–Mortici–Marinescu Problem. This problem, going back to an idea of M.S. Klamkin and succively extended (see[2], [7] and [6]), is the following:



Show that for any point z inside or outside a parallelogram a, b, c, d we have

$$|az \cdot cz - bz \cdot dz| \leq ba \cdot da \leq az \cdot cz + bz \cdot dz \tag{5}$$

(see Figure (c)).

Proof. This time we write equations (3) and (4), which together with (2) also remain valid for parallelograms, as

$$(a - z)(c - z) - (b - z)(d - z) = -(b - a)(d - a)\,. \tag{6}$$

From this, the inequalities (5) follow immediately by taking absolute values and using the triangle inequality.

But we want to understand a little more: *are the inequalities in* (5) *sharp and, if "yes", for which points z?* To answer this, we place the parallelogram with the mid-point in the origin and rotate until the point a is lowest and the angle bisectors bad, adc, etc. are parallel to the axes. Then the complex number $-(b - a)(d - a)$ becomes real and positive. This vector connects for all z the vectors $(a - z)(c - z)$ and $(b - z)(d - z)$. It is clear that for optimality we must have that $(a - z)(c - z)$, and thus also $(b - z)(d - z)$, are real. This means that the two angles α and γ in Figure (c) must be equal. This is equivalent with the fact that z lies on the equilateral hyperbola through the four points a, b, c, d (dotted in Fig. (c); this last property, if not known since Apollonius, is seen by the identity $(u - \frac{1}{v})/(v - \frac{1}{u}) = (u + \frac{1}{v})/(v + \frac{1}{u})$). Inside the parallelogram the right inequality is sharp, outside the parallelogram the left one.

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