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An extension of the Lobachevsky formula

Hassan Jolany

Hassan Jolany received his Ph.D. under the direction of Gang Tian (Princeton University) from the Université Lille 1 in June 2016. He then had a lecturer position at the University of California, San Diego for one quarter in fall 2016. He visited Princeton University in 2015. His main fields of research are analytical algebraic geometry and geometric flows.

1 Introduction

The Dirichlet integral plays an important role in distribution theory. We can see the Dirichlet integral in terms of distribution. The following classical Dirichlet integral has drawn lots of attention:

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

We can use the theory of residues to evaluate this Dirichlet integral formula. G.H. Hardy and A.C. Dixon gave a lot of different proofs for it. See [5–7]. In this paper we give an elegant method to generalize this Lobachevsky formula.

We start with the following elementary lemma. See [1–4].

Lemma 1.1. *For $\alpha \notin \mathbb{Z}\pi$, we have*

$$\frac{1}{\sin\alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{\alpha - m\pi} + \frac{1}{\alpha + m\pi} \right).$$

Dirichlet-Integrale, wie etwa das uneigentliche Integral $\int_0^\infty \frac{\sin(x)}{x} dx$ des Sinus cardinalis, tauchen an verschiedenen Stellen der Analysis auf: beispielsweise bei der Fourier-Transformation, in der Theorie der Distributionen, bei der Laplace-Transformation oder in der Signaltheorie. Derartige Integrale wurden unter anderem von Godfrey Harold Hardy und Alfred Cardew Dixon untersucht. Von Nikolai Lobatschewski stammt eine erstaunliche Formel für den Fall eines Integranden, der das Produkt der sinc-Funktion und einer π-periodischen Funktion ist. Der Autor der vorliegenden Arbeit erweitert nun die Formel von Lobatschewski auf Integranden, welche das Produkt höherer Potenzen von sinc und einer π-periodischen Funktion sind.

Proof. For every positive integer N, denote by C_N the positively-oriented square in the complex plane with vertices $(N+\frac{1}{2})(\pm 1 \pm i)$. On the one hand, since the function $1/\sin(z)$ is bounded on C_N by a constant which is independent of N, one has

$$\oint_{C_N} \frac{2\pi\alpha dz}{(z^2\pi^2-\alpha^2)\sin(\pi z)} \to 0$$

as $N \to \infty$. On the other hand, by the Residue Theorem, one also gets

$$\oint_{C_N} \frac{2\pi\alpha dz}{(z^2\pi^2-\alpha^2)\sin(\pi z)} = \sum_{n=-N}^{N} (-1)^n \frac{2\alpha}{n^2\pi^2-\alpha^2} + \frac{1}{\sin(\alpha)}$$

which proves the claim as $N \to \infty$. □

Lemma 1.2. *For $\alpha \notin \mathbb{Z}\pi$, we have the identity*

$$\frac{1}{\sin^2\alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty}\left(\frac{1}{(\alpha-m\pi)^2} + \frac{1}{(\alpha+m\pi)^2}\right).$$

Proof. The identity follows by differentiating termwise the classical formula,

$$\cot(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{2z}{z^2-m^2\pi^2}.$$

□

2 The Lobachevsky formula

Now, we present the Lobachevsky formula.

Theorem 2.1. *Let $f(x)$ be a continuous function which satisfies $f(x+\pi) = f(x)$, and $f(\pi - x) = f(x)$, $0 \le x < \infty$. If the the following integrals exist in the sense of the improper Riemann integrals, then we have the following Lobachevsky identity*

$$\int_0^\infty \frac{\sin^2 x}{x^2} f(x)dx = \int_0^\infty \frac{\sin x}{x} f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx.$$

Proof. Take

$$I = \int_0^\infty \frac{\sin x}{x} f(x)dx;$$

then we can write I as

$$I = \sum_{\upsilon=0}^{\infty} \int_{\upsilon\frac{\pi}{2}}^{(\upsilon+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x)dx,$$

where $\upsilon = 2\mu - 1$ or $\upsilon = 2\mu$. By changing $x = \mu\pi + t$ or $x = \mu\pi - t$ we get

$$\int_{2\mu\frac{\pi}{2}}^{(2\mu+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^\mu \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi + t} f(t)dt$$

and
$$\int_{(2\mu-1)\frac{\pi}{2}}^{(2\mu)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu-1} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi - t} f(t)dt;$$
so we get
$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} f(t)dt + \sum_{\mu=1}^{\infty} \int_0^{\frac{\pi}{2}} (-1)^{\mu} f(t) \left(\frac{1}{t+\mu\pi} + \frac{1}{t-\mu\pi} \right) \sin t dt.$$

Consequently we can write I in the form
$$I = \int_0^{\frac{\pi}{2}} \sin t \left(\frac{1}{t} + \sum_{\mu=1}^{\infty} (-1)^{\mu} \left(\frac{1}{t+\mu\pi} + \frac{1}{t-\mu\pi} \right) \right) f(t)dt.$$

Hence
$$I = \int_0^{\frac{\pi}{2}} f(t)dt$$
and the proof of the identity $\int_0^{\infty} \frac{\sin x}{x} f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx$ is complete. Now we prove the second part of the identity. Taking
$$J = \int_0^{\infty} \frac{\sin^2 x}{x^2} f(x)dx,$$
we can write J as
$$J = \sum_{\upsilon=0}^{\infty} \int_{\upsilon\frac{\pi}{2}}^{(\upsilon+1)\frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x)dx,$$
where $\upsilon = 2\mu - 1$ or $\upsilon = 2\mu$. By changing $x = \mu\pi + t$ or $x = \mu\pi - t$ we get
$$\int_{2\mu\frac{\pi}{2}}^{(2\mu+1)\frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x) = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{(\mu\pi + t)^2} f(t)dt$$
and
$$\int_{(2\mu-1)\frac{\pi}{2}}^{(2\mu)\frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x) = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{(\mu\pi - t)^2} f(t)dt;$$
so we get
$$J = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{t^2} f(t)dt + \sum_{\mu=1}^{\infty} \int_0^{\frac{\pi}{2}} f(t) \left(\frac{1}{(t+\mu\pi)^2} + \frac{1}{(t-\mu\pi)^2} \right) \sin^2 t dt.$$

Consequently we can write J in form
$$J = \int_0^{\frac{\pi}{2}} \sin^2 t \left(\frac{1}{t^2} + \sum_{\mu=1}^{\infty} \left(\frac{1}{(t+\mu\pi)^2} + \frac{1}{(t-\mu\pi)^2} \right) \right) f(t)dt.$$

Hence from Lemma 1.2, we get

$$J = \int_0^{\frac{\pi}{2}} f(t)dt$$

and the proof is complete. □

3 Extension of the Lobachevsky formula

Now we give a general method for calculating the following Dirichlet integral:

$$\int_0^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x)dx,$$

where $f(\pi + x) = f(x)$, and $f(\pi - x) = f(x)$, $0 \le x < \infty$. Here we have assumed that f is continuous and $\int_0^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x)dx$ is defined in the sense of the improper Riemann integral. We start with $n = 2$. As we did in the previous section, take

$$I = \int_0^{\infty} \frac{\sin^4 x}{x^4} f(x)dx.$$

By a direct computation

$$\frac{d^2}{dx^2}\left(\frac{1}{\sin^2 x}\right) = \frac{6}{\sin^4(x)} - \frac{4}{\sin^2(x)}.$$

Next, differentiating twice termwise the right-hand side of the identity of Lemma 1.2, we get the identity

$$\frac{1}{\sin^4 \alpha} - \frac{2}{3\sin^2 \alpha} = \frac{1}{\alpha^4} + \sum_{m=1}^{\infty}\left(\frac{1}{(\alpha - m\pi)^4} + \frac{1}{(\alpha + m\pi)^4}\right).$$

From the previous method which we explained in Section 2, we can write I as

$$I = \int_0^{\frac{\pi}{2}} \sin^4 t \left(\frac{1}{\sin^4 t} - \frac{2}{3\sin^2 t}\right) f(t)dt.$$

Hence

$$I = \int_0^{\frac{\pi}{2}} f(t)dt - \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin^2 t f(t)dt.$$

So we proved the following theorem:

Theorem 3.1. *Let f satisfy $f(x + \pi) = f(x)$, and $f(\pi - x) = f(x)$, $0 \le x < \infty$. If the integral*

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} f(x)dx$$

is defined in the sense of the improper Riemann integral, then we have the equality

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} f(x)dx = \int_0^{\frac{\pi}{2}} f(t)dt - \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin^2 t f(t)dt.$$

In particular, if we take $f(x) = 1$ in the previous theorem, we obtain:

Remark 3.2. We have
$$\int_0^\infty \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}.$$

We have also the following remark from the Lobachevsky formula:

Remark 3.3. If $f(x)$ satisfies the condition $f(x+\pi) = f(x)$, and $f(\pi - x) = f(x)$, $0 \le x < \infty$, take
$$I = \int_0^\infty \frac{\sin^{2n+1} x}{x} f(x)dx = \int_0^\infty \sin^{2n} x \frac{\sin x}{x} f(x)dx.$$

If we set $\sin^{2n} x f(x) = g(x)$, we get $g(x+\pi) = g(x)$, $g(\pi - x) = g(x)$, now if we take $f(x) = 1$, then
$$\int_0^\infty \frac{\sin^{2n+1} x}{x} dx = \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}.$$

Now, by the following important remark, we can calculate the Lobachevsky formula for any $n \ge 3$. Let $f(z)$ satisfy the conditions of the beginning of Section 3.

Remark 3.4. In fact, the Dirichlet integral
$$\int_0^\infty \frac{\sin^{2n} z}{z^{2n}} f(z)dz$$
has the form (for $n \ge 3$)
$$\alpha_1 \int_0^{\frac{\pi}{2}} f(z)dz + \alpha_2 \int_0^{\frac{\pi}{2}} \cot^{2n-2}(z) \sin^{2n-2}(z) f(z)dz + \cdots + \alpha_k \int_0^{\frac{\pi}{2}} \cot^2(z) \sin^2(z) f(z)dz,$$
where the constants α_i can be computed by the use of the following formulas (and the help of the engine Wolfram Alpha for instance): For every positive integer n, one can compute
$$\frac{d^n}{dz^n}\left(\frac{1}{\sin^2(z)}\right) = \frac{d^n}{dz^n}(1+\cot^2 z) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dz^{n-k}}(\cot z) \frac{d^k}{dz^k}(\cot z)$$
by the Leibnitz rule and then apply the closed formula
$$\frac{d^m}{dz^m}(\cot z) = (2i)^m (\cot(z) - i) \sum_{j=1}^{m} \left\{ {m \atop k} \right\} (i \cot(z) - 1)^k$$
of Lemma 2.1. of [8], where $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind. Now by applying the identity
$$\frac{d^n}{dz^n}\left(\frac{1}{\sin^2 z}\right) = \sum_{k=-\infty}^{\infty} \frac{(-1)^n (n+1)!}{(z+k\pi)^{n+2}}$$

we can find a closed formula for such a Dirichlet integral for any n. For example, when $n = 3$ we have $\alpha_1 = \frac{2}{15}, \alpha_2 = \frac{2}{15}, \alpha_3 = \frac{11}{15}$ and for $n = 4$, $\alpha_1 = \frac{272}{7!}, \alpha_2 = \frac{64}{7!}, \alpha_3 = \frac{1824}{7!}, \alpha_4 = \frac{2880}{7!}$.

Remark 3.5. We have the following formulas:

1) $\displaystyle\int_0^\infty \frac{\sin^6 z}{z^6} dz = \frac{11\pi}{40}$

2) $\displaystyle\int_0^\infty \frac{\sin^8 z}{z^8} dz = \frac{151\pi}{630}$

3) $\displaystyle\int_0^\infty \frac{\sin^{10} z}{z^{10}} dz = \frac{15619\pi}{72576}$

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Hassan Jolany
University of Lille 1
Lille, France
e-mail: hassan.jolany@math.univ-lille1.fr