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A few simple Levi-Civita functional equations on groups

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1 Introduction

Let G be a group, S be a semi-group, and K be a field. Specifically, the field of complex numbers will be denoted by $\mathbb{C}$. The identity element of a group and semi-group will be

[1]The authors would like to dedicate this paper to the memory of Prasanna K. Sahoo, who passed away June 18, 2017. He played an integral part in the research and formation of this paper.

Funktionalgleichungen tauchen in der Mathematik und der mathematischen Physik regelmässig auf: Schon Cauchy zeigte, dass die stetigen Lösungen $f : \mathbb{R} \to \mathbb{R}$ der Gleichung $f(x + y) = f(x) + f(y)$ die linearen Funktionen $f(x) = ax$ sind. Multiplikativ geschrieben, $f(xy) = f(x)f(y)$ entsteht die Funktionalgleichung der Potenzfunktionen $f(x) = x^a$, $f(x + y) = f(x)f(y)$ liefert die Exponentialfunktionen $f(x) = a^x$ und $f(xy) = f(x) + f(y)$ die Logarithmusfunktionen. Die Additionstheoreme von trigonometrischen und hyperbolischen Funktionen (insbesondere die relativistische Addition der Geschwindigkeit), die Funktionalgleichungen der Gamma-Funktion oder der Bernoulli-Polynome sind weitere Beispiele. In der vorliegenden Arbeit wird die Levi-Civita-Gleichung und deren symmetrisierte Variante, insbesondere $f(xy) + g(yx) = 2\,h(x)\,k(y)$, in ganz allgemeinem Rahmen studiert.

denoted by e. A function $f : G \to K$ is said to be an *additive function* if it satisfies the functional equation

$$f(xy) = f(x) + f(y)$$

for all $x, y \in G$. If G is an arbitrary group, then the group operation will be denoted by multiplication. Thus, a function $f : G \to K$ that satisfies the functional equation

$$f(xy) = f(x)f(y)$$

for all $x, y \in G$ is said to be a *multiplicative function* or a *multiplicative homomorphism*. A function $f : G \to K$ is called a *central function* if and only if $f(xy) = f(yx)$ for all $x, y \in G$.

In 1913 Levi-Civita in [4] began studying the functional equation

$$f(xy) = \sum_{i=1}^{n} g_i(x)h_i(y),$$

where $f, g_i, h_i : G \to \mathbb{C}$ for all $1 \leq i \leq n$, known as the Levi-Civita functional equation. One can easily see that in the case that $n = 2$ and $g_1 = g_2 = f$ and $h_2 = h_1 = 1$ it is a generalization of the previously mentioned additive and multiplicative functions dependent upon the operation. Also when written additively, it becomes the addition formula for the sine and cosine functions when $n = 2$. Along with these, it encompasses several other interesting functional equations such as the following:

$$\begin{aligned} f(xy) &= f(x)\, f(y) + f(y) \\ f(xy) &= 2\, f(x) + \, 2\, f(y) \\ f(xy) &= 2\, f(x) + \, 2\, f(y) + \lambda\, f(x)\, f(y) \end{aligned}$$

where $f : G \to \mathbb{C}$. These functional equations have been widely studied along with several generalizations. The most popular generalizations are forms of the functional equation

$$f(xy) + f(yx) = \sum_{i=1}^{n} g_i(x)h_i(y)$$

where $f, g_i, h_i : G \to \mathbb{C}$ for all $1 \leq i \leq n$. It is known as the symmetrized Levi-Civita functional equation. In the case that $n = 2$, $h_1 = g_2$, and $g_1 = h_2 = f$ one gets what is known as the symmetrized sine functional equation, a generalization of the original sine functional equation.

2 Preliminary results

Several preliminary results relating to the symmetrized Levi-Civita functional equation will be useful for us throughout this paper. Many simple forms of the symmetrized Levi-Civita functional equation have been widely studied, most notably the functional equations

$$\begin{aligned} f(xy) + f(yx) &= 2\, f(x) + 2\, f(y) \\ f(xy) + f(yx) &= 2\, f(x)\, f(y) \end{aligned} \tag{2.1}$$

for all $x, y \in G$. They are often referred to as the additive and multiplicative Cauchy equations. The solution to the multiplicative Cauchy functional equation was first posed as an open problem by Corovei: if f is a solution to (2.2), must f be a multiplicative function? Radó was one of the many to attempt Corovei's question. He was in fact the first to conclude in [6] that yes, f must be a multiplicative function. His proof was later shortened by Pl. Kannappan in [2]. Stetkær later studied the equation defined on a semi-group in [5] giving us the theorem below.

Theorem 1. *Let $f : S \to \mathbb{C}$ be a complex function on a semi-group S satisfying the symmetrized multiplicative Cauchy equation*

$$\frac{f(xy) + f(yx)}{2} = f(x)\, f(y) \tag{2.2}$$

for all $x, y \in S$. Then $f(xy) = f(yx)$ for all $x, y \in S$, and so $f : S \to \mathbb{C}$ is a multiplicative function.

In this paper we consider a few generalizations of the symmetrized Levi-Civita functional equation based upon the results of Corovei, Radó, Kannappan, and Stetkær. We work towards generalizing some of the previously mentioned addition formulas on an arbitrary group. If one were to consider looking at a sine function or a generalization of one on a non-abelian group, it is the hope that these equations would provide some insight into a different way to go about that. Most notably we consider the following functional equation defined on a semi-group S:

$$f(xy) + g(yx) = 2\, h(x)\, k(x). \tag{2.3}$$

It leads us to another generalization that has evolved over time; that being the equation

$$f(xy) + f(yx) = 2\, f(x) + 2\, f(y) + 2\, \lambda\, f(x) f(y).$$

This equation has roots dating back to an equation studied before the now known Levi-Civita equation, an equation studied by Jansen in 1878 ([5], see exercise 3.7); that is

$$f(x + y) = f(x) + f(y) + f(x) f(y)$$

where the unknown function f is a continuous, complex-valued function defined on an interval on the real line.

3 Several generalizations along with a combination equation

We first consider several generalizations of the symmetrized multiplicative Cauchy functional equation (2.2). We start with a generalization in terms of two functions f and g.

Theorem 2. *Let S be a semi-group and $\mathbb{C}$ be the field of complex numbers. If the non-zero functions $f, g : S \to \mathbb{C}$ satisfy the functional equation*

$$f(xy) + f(yx) = g(x) f(y), \tag{3.1}$$

then

$$f(x) = \alpha\, \theta(x) \qquad g(x) = 2\, \theta(x),$$

where θ is a multiplicative function and $\alpha \in \mathbb{C}$ is an arbitrary non-zero complex constant.

Proof. Let $y = e$ in (3.1), then

$$2\,f(x) = g(x) f(e)$$

for all $x \in S$. Since f, g are both non-zero, we have that $f(e) \neq 0$. Therefore, letting $f(e) = \alpha \in \mathbb{C}$ we get that

$$f(x) = \frac{\alpha}{2}\,g(x)$$

for all $x \in S$. Substituting this back into (3.1) and dividing by $\frac{\alpha}{2}$ we get the following:

$$g(xy) + g(yx) = g(x)\,g(y).$$

Multiplying through by $\frac{1}{2}$ gives us

$$\frac{g(xy)}{2} + \frac{g(yx)}{2} = 2\left(\frac{g(x)}{2}\right)\left(\frac{g(y)}{2}\right)$$

for all $x, y \in S$. Therefore, from Theorem 1 we have that

$$g(x) = 2\,\theta(x) \qquad f(x) = \alpha\,\theta(x)$$

where θ is a multiplicative function and $\alpha \in \mathbb{C}$ is an arbitrary non-zero complex constant. This completes the proof. □

One can see switching x and y in the previous theorem takes care of the case $f(xy) + f(yx) = f(x)g(y)$. Hence we move on to another generalization in terms of two functions.

Theorem 3. *Let S be a semi-group and $\mathbb{C}$ be the field of complex numbers. If the non-zero functions $f, g : S \to \mathbb{C}$ satisfy the functional equation*

$$f(xy) + f(yx) = g(x)g(y), \tag{3.2}$$

then

$$f(x) = \frac{\alpha^2}{2}\,\theta(x) \qquad g(x) = \alpha\,\theta(x)$$

where θ is a multiplicative function and $\alpha \in \mathbb{C}$ is an arbitrary non-zero complex constant.

Proof. Let $y = e$ in (3.2), then

$$2\,f(x) = g(x)g(e)$$

for all $x \in S$. Since f, g are non-zero, we get that $g(e) \neq 0$. Therefore, letting $g(e) = \alpha \in \mathbb{C}$, we get that

$$f(x) = \frac{\alpha}{2}\,g(x)$$

for all $x \in S$. Substituting this into (3.2) and multiplying through by $\frac{2}{\alpha^2}$ we get the following:

$$\frac{1}{\alpha}\,g(xy) + \frac{1}{\alpha}\,g(yx) = 2\left(\frac{1}{\alpha}\,g(x)\right)\left(\frac{1}{\alpha}\,g(y)\right)$$

for all $x, y \in S$. Therefore, from Theorem 1 we have

$$g(x) = \alpha\,\theta(x) \qquad f(x) = \frac{\alpha^2}{2}\,\theta(x)$$

where θ is a multiplicative function and $\alpha \in \mathbb{C}$ is an arbitrary non-zero complex constant. This completes the proof. □

Next, we expand our generalizations to three functions, f, g, and h.

Theorem 4. *Let S be a semi-group and $\mathbb{C}$ be the field of complex numbers. If the non-zero functions $f, g, h : S \to \mathbb{C}$ satisfy the functional equation*

$$f(xy) + g(yx) = h(x)h(y), \tag{3.3}$$

then

$$\begin{aligned} f(x) &= \frac{\alpha^2}{2}\,\theta(x) + \frac{1}{2}\gamma(x) \\ g(x) &= \frac{\alpha^2}{2}\,\theta(x) - \frac{1}{2}\gamma(x) \\ h(x) &= \alpha\,\theta(x) \end{aligned}$$

where θ is a multiplicative function, $\alpha \in \mathbb{C}$ is an arbitrary non-zero complex constant, and γ is a central function.

Proof. Switching x and y in (3.3) yields

$$f(yx) + g(xy) = h(x)h(y)$$

for all $x, y \in S$. Subtracting this from (3.3) and rearranging gives us the following:

$$f(xy) - g(xy) = f(yx) - g(yx)$$

for all $x, y \in S$. Defining $\gamma : S \to \mathbb{C}$ such that

$$\gamma(x) = f(x) - g(x),$$

we get that

$$\gamma(xy) = \gamma(yx).$$

Therefore, γ is a central function.

Setting $y = e$ in the original equation we get

$$f(x) + g(x) = h(x)h(e)$$

for all x in S. Let $h(e) = \alpha$, a complex constant in $\mathbb{C}$. We consider two cases, $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ we have that

$$f(x) = -g(x)$$

for all $x \in S$. Substituting this into (3.3) we have that

$$-g(yx) + g(xy) = h(x)h(y)$$

for all $x, y \in S$. Switching x and y and adding to the previous equation we get the following:

$$0 = 2\,h(x)h(y)$$

for all $x, y \in S$. Since f, g, h are non-zero functions, this is not possible. Therefore, $\alpha \neq 0$.

Now, switching x and y in (3.3) and adding yields:

$$f(xy) + g(xy) + f(yx) + g(yx) = 2\,h(x)\,h(y) \tag{3.4}$$

for all $x, y \in S$. Using the fact that

$$f(x) + g(x) = \alpha\,h(x)$$

and multiplying through by $\frac{1}{\alpha^2}$, (3.4) becomes

$$\frac{1}{\alpha}h(xy) + \frac{1}{\alpha}h(yx) = 2\left(\frac{1}{\alpha}\,h(x)\right)\left(\frac{1}{\alpha}\,h(y)\right)$$

for all $x, y \in S$. Therefore, from Theorem 1 we get

$$h(x) = \alpha\,\theta(x)$$

where θ is a multiplicative function and $\alpha \in \mathbb{C}$ is an arbitrary non-zero complex constant. Hence

$$f(x) + g(x) = \alpha\,h(x) = \alpha^2\,\theta(x)$$

for all $x \in S$. Given that

$$f(x) - g(x) = \gamma(x),$$

adding and subtracting the previous two functional equations yields

$$\begin{aligned} f(x) &= \frac{\alpha^2}{2}\,\theta(x) + \frac{1}{2}\gamma(x) \\ g(x) &= \frac{\alpha^2}{2}\,\theta(x) - \frac{1}{2}\gamma(x) \\ h(x) &= \alpha\,\theta(x) \end{aligned}$$

where θ is a multiplicative function, $\alpha \in \mathbb{C}$ is an arbitrary non-zero complex constant, and γ is a central function. This completes the proof. □

Lastly, we generalize (2.2) to four non-zero functions.

Theorem 5. *Let G be a group and K a field. If the non-zero functions $f, g, h, k : G \to K$ satisfy the functional equation*

$$f(xy) + g(yx) = h(x)k(y), \tag{3.5}$$

then

$$\begin{aligned} f(x) &= \alpha\,\beta\,\frac{\mu^2}{2}\,\theta(x) + \frac{1}{2}\gamma\,(x) \\ g(x) &= \alpha\,\beta\,\frac{\mu^2}{2}\,\theta(x) - \frac{1}{2}\gamma\,(x) \\ h(x) &= \alpha\,\mu\,\theta(x) \\ k(x) &= \beta\,\mu\,\theta(x) \end{aligned}$$

where θ is a multiplicative function, $\alpha, \beta, \mu \in K$ are arbitrary non-zero constants, and γ is a central function.

Proof. Let $y = e$ in (3.5), then

$$f(x) + g(x) = h(x)k(e)$$

for all $x \in G$. We consider two cases, $k(e) = 0$ and $k(e) \neq 0$. If $k(e) = 0$, then we have that

$$f(x) + g(x) = 0, \qquad g(x) = -f(x)$$

for all $x \in G$. Therefore,

$$f(xy) - f(yx) = h(x)k(y)$$

for all $x, y \in G$. Setting $y = x^{-1}$ gives us

$$\begin{aligned} f(x\,x^{-1}) - f(x^{-1}x) &= h(x)k(x^{-1}) \\ 0 &= h(x)k(x^{-1}) \end{aligned}$$

which implies either $h = 0$ or $k = 0$. This is not possible since f, g, h, k are non-zero functions. Therefore, $k(e) \neq 0$. One can easily show that $h(e) \neq 0$.

Let $h(e) = \alpha$ and $k(e) = \beta$ where α and β are non-zero constants in K. If $y = e$ in (3.5) we have

$$\begin{aligned} f(x) + g(x) &= \beta\,h(x) \\ \frac{1}{\beta}(f(x) + g(x)) &= h(x) \end{aligned} \tag{3.6}$$

for all $x \in G$. Similarly if $x = e$ in (3.5), we have

$$\begin{aligned} f(y) + g(y) &= \alpha\,k(x) \\ \frac{1}{\alpha}(f(x) + g(x)) &= k(x) \end{aligned} \tag{3.7}$$

for all $y \in G$. Therefore, rewriting our original equation we have that

$$f(xy) + g(yx) = \frac{1}{\alpha\beta}(f(x) + g(x))\,(f(x) + g(x))$$
$$\frac{1}{\alpha\beta} f(xy) + \frac{1}{\alpha\beta} g(yx) = \left(\frac{1}{\alpha\beta}(f(x) + g(x))\right)\left(\frac{1}{\alpha\beta}(f(x) + g(x))\right)$$

for all $x, y \in G$. From Theorem 4 we have that

$$f(x) = \alpha\beta\frac{\mu^2}{2}\theta(x) + \frac{1}{2}\gamma(x)$$
$$g(x) = \alpha\beta\frac{\mu^2}{2}\theta(x) - \frac{1}{2}\gamma(x)$$
$$f(x) + g(x) = \alpha\beta\mu\,\theta(x)$$

where θ is a multiplicative function, $\mu \in K$ is an arbitrary non-zero constant, and γ is a central function. Therefore, using (3.6) and (3.7) we get the following:

$$h(x) = \frac{1}{\beta}\alpha\beta\mu\,\theta(x) = \alpha\mu\,\theta(x), \qquad k(x) = \frac{1}{\alpha}\alpha\beta\mu\,\theta(x) = \beta\mu\,\theta(x)$$

for all $x \in G$. This completes the proof. □

We now give a concrete example to the generalization in terms of four functions, that is equation (3.5).

Example 6. *Consider the non-abelian group of* 2×2 *invertible matrices over* $\mathbb{R}$*,*

$$GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}.$$

Consider the determinate function $\det : GL_2(\mathbb{R}) \to \mathbb{R}$ *defined such that* $\det(A) = ad - bc$ *for all* $A \in GL_2(\mathbb{R})$ *and the trace function* $\mathrm{tr} : GL_2(\mathbb{R}) \to \mathbb{R}$ *defined such that* $\mathrm{tr}(A) = \mathrm{trace}(A) = a + d$ *for all* $A \in GL_2(\mathbb{R})$*. It is known that the following hold for all* $A, B \in GL_2(\mathbb{R})$*:*

$$\det(A\,B) = \det(A)\,\det(B) = \det(B)\,\det(A) = \det(B\,A)$$
$$\mathrm{tr}(A\,B) = \mathrm{tr}(B\,A).$$

Therefore, det *is a multiplicative function that is also central and* tr *is a central function. One can easily show that*

$$f(x) = \alpha\beta\frac{\mu^2}{2}\det(x) + \frac{1}{2}\mathrm{tr}(x)$$
$$g(x) = \alpha\beta\frac{\mu^2}{2}\det(x) - \frac{1}{2}\mathrm{tr}(x)$$
$$h(x) = \alpha\mu\,\det(x)$$
$$k(x) = \beta\mu\,\det(x)$$

where $\alpha, \beta, \mu \in \mathbb{R}$ *is a solution to the equation* (3.5).

Now, we will consider a bit more complex Levi-Civita functional equation and a few generalizations of it.

Theorem 7. *Let G be a group and K a field with* $\mathrm{char}(K) \neq 2$ *and $\lambda \in K$ a non-zero a priori chosen constant. If $f, g : G \to K$ satisfy the functional equation*

$$f(xy) + f(yx) = 2\,g(x) + 2\,g(y) + 2\,\lambda\,g(x)\,g(y) \tag{3.8}$$

for all $x, y \in G$, then

$$f(x) = \frac{\alpha^2\,\phi(x) - 1}{\lambda}, \qquad g(x) = \frac{\alpha\,\phi(x) - 1}{\lambda}$$

where ϕ is a multiplicative function and α and β are non-zero constants in K.

Proof. We start by multiplying through by λ to get

$$\lambda\,f(xy) + \lambda\,f(yx) = 2\,\lambda\,g(x) + 2\,\lambda\,g(y) + 2\,\lambda^2\,g(x)\,g(y).$$

Adding 2 to both sides and rewriting gives us the following:

$$[1 + \lambda\,f(xy)] + [1 + \lambda\,f(yx)] = 2\,[1 + \lambda\,g(x) + \lambda\,g(y) + \lambda^2\,g(x)\,g(y)]$$
$$[1 + \lambda\,f(xy)] + [1 + \lambda\,f(yx)] = 2\,[1 + \lambda\,g(x)][1 + \lambda\,g(y)]. \tag{3.9}$$

Define $\omega : G \to K$ and $\psi : G \to K$ such that

$$\omega(x) = 1 + \lambda\,f(x) \tag{3.10}$$
$$\psi(x) = 1 + \lambda\,g(x) \tag{3.11}$$

for all $x \in G$. Using (3.10) and (3.11) we can rewrite (3.9) giving us

$$\omega(xy) + \omega(yx) = 2\,\psi(x)\,\psi(y). \tag{3.12}$$

With $y = e$ the previous equation becomes

$$2\,\omega(x) = 2\,\psi(x)\,\psi(e).$$

Since $\psi(e)$ is a constant, we define α such that $\alpha = \psi(e)$. Thus

$$\omega(x) = \psi(x)\,\alpha. \tag{3.13}$$

Now, since α is a constant we must consider two cases: $\alpha = 0$ and $\alpha \neq 0$. In the case that $\alpha = 0$, using (3.13) we get $\omega(x) = 0$ for all $x \in G$. By this fact and (3.12)

$$0 = 2\,\psi(x)\,\psi(y),$$

which implies that $\psi(x) = 0$ for all $x \in G$. Thus, since $\omega(x) = 0$ and $\psi(x) = 0$ we can go back and solve (3.10) and (3.11) for f and g respectively.

$$f(x) = -\frac{1}{\lambda} \qquad \text{and} \qquad g(x) = -\frac{1}{\lambda}$$

which are solutions to (3.8) included in the case in which $\lambda \neq 0$.

In the case that $\alpha \neq 0$, using (3.13) we can rewrite (3.12)

$$\psi(xy)\,\alpha + \psi(yx)\,\alpha = 2\,\psi(x)\,\psi(y)$$
$$\psi(xy) + \psi(yx) = \frac{2}{\alpha}\,\psi(x)\,\psi(y).$$

Since $\alpha \neq 0$, we can divide through by α to get the following:

$$\frac{\psi(xy) + \psi(yx)}{\alpha} = \frac{2}{\alpha^2}\,\psi(x)\,\psi(y)$$
$$\frac{\psi(xy)}{\alpha} + \frac{\psi(yx)}{\alpha} = 2\left(\frac{\psi(x)}{\alpha}\right)\left(\frac{\psi(y)}{\alpha}\right). \tag{3.14}$$

Define $\phi : G \to K$ such that

$$\phi(x) = \frac{\psi(x)}{\alpha} \tag{3.15}$$

for all $x \in G$. Thus, (3.14) becomes

$$\phi(xy) + \phi(yx) = 2\,\phi(x)\,\phi(y).$$

By [2] we have that ϕ is a multiplicative function. Now, (3.15) gives us

$$\alpha\,\phi(x) = \psi(x). \tag{3.16}$$

Substituting this into (3.13) yields

$$\omega(x) = \alpha^2\,\phi(x).$$

Comparing this with (3.10) we get

$$\alpha^2\,\phi(x) = 1 + \lambda\,f(x)$$
$$f(x) = \frac{\alpha^2\,\phi(x) - 1}{\lambda}.$$

Similarly, comparing (3.16) with (3.11) gives us

$$\alpha\,\phi(x) = 1 + \lambda\,g(x)$$
$$g(x) = \frac{\alpha\,\phi(x) - 1}{\lambda}.$$

Hence, we have the final solutions of (3.8), which completes the proof. □

If we let $f = g$ in the previous theorem, we have the following corollary.

Corollary 8. *Let G be a group, K be a field of characteristic different from 2, and $\lambda \in K$ be a non-zero a priori chosen constant. The non-constant function $f : G \to K$ satisfies the functional equation*

$$f(xy) + f(yx) = 2\,f(x) + 2\,f(y) + 2\,\lambda\,f(x)\,f(y) \tag{3.17}$$

for all $x, y \in G$ if and only if for all $x \in G$

$$f(x) = \frac{\phi(x) - 1}{\lambda}$$

where ϕ is a multiplicative function.

Proof. Letting $f = g$ in Theorem 7 we have the following:

$$\begin{aligned} f(x) = \frac{\alpha^2\,\phi(x) - 1}{\lambda} &= \frac{\alpha\,\phi(x) - 1}{\lambda} \\ \alpha^2\,\phi(x) &= \alpha\,\phi(x) \\ \alpha\,\phi(x)(\alpha - 1) &= 0 \end{aligned}$$

for all $x \in K$, where θ, ϕ are multiplicative functions, and α is a non-zero constant in K. Since f is non-constant we have that $\phi \neq 0$ and $\alpha \neq 0$. Therefore, $\alpha = 1$ and

$$f(x) = \frac{\alpha\,\phi(x) - 1}{\lambda}$$

for all $x \in G$, which completes the proof. □

Theorem 9. *Let G be a group, K be a field with* char$(K) \neq 2$*, and $\lambda \in K$ be a non-zero a priori chosen constant. If $f, g, h : G \to K$ satisfy the functional equation*

$$f(xy) + g(yx) = 2\,h(x) + 2\,h(y) + 2\,\lambda\,h(x)\,h(y) \tag{3.18}$$

for all $x, y \in G$, then

$$\begin{aligned} f(x) &= \frac{\alpha^2\phi(x) - 1}{\lambda} + \frac{\gamma(x)}{2} \\ g(x) &= \frac{\alpha^2\phi(x) - 1}{\lambda} - \frac{\gamma(x)}{2} \\ h(x) &= \frac{\alpha\phi(x) - 1}{\lambda} \end{aligned}$$

for all $x \in G$, where ϕ is a multiplicative function, $\gamma : G \to K$ is a central function, and α is a non-zero constant in K.

Proof. Interchanging x and y in (3.18) yields

$$f(yx) + g(xy) = 2\,h(y) + 2\,h(x) + 2\,\lambda\,h(y)\,h(x). \tag{3.19}$$

Subtracting the previous equation from (3.18)

$$\begin{aligned} f(xy) + g(yx) - f(yx) - g(xy) &= 0 \\ (f - g)(xy) - (f - g)(yx) &= 0. \end{aligned} \tag{3.20}$$

Defining $\gamma : G \to K$ such that

$$\gamma(x) = (f - g)(x), \tag{3.21}$$

then (3.20) reduces to

$$\gamma(xy) = \gamma(yx).$$

Therefore, γ is a central function.
Now, adding (3.18) and (3.19) we find that

$$f(xy) + g(yx) + f(yx) + g(xy) = 4\,h(x) + 4\,h(y) + 4\,\lambda\,h(x)\,h(y)$$
$$\frac{(f+g)(xy)}{2} + \frac{(f+g)(yx)}{2} = 2\,h(x) + 2\,h(y) + 2\,\lambda\,h(x)\,h(y). \tag{3.22}$$

Defining $\psi : G \to K$ such that

$$\psi(x) = \frac{(f+g)(x)}{2},$$

reduces (3.22) to

$$\psi(xy) + \psi(yx) = 2\,h(x) + 2\,h(y) + 2\,\lambda\,h(x)\,h(y).$$

From Theorem 7 we get

$$\psi(x) = \frac{\alpha^2\phi(x) - 1}{\lambda}$$
$$h(x) = \frac{\alpha\phi(x) - 1}{\lambda},$$

where ϕ is a multiplicative function and α is a non-zero constant in K. This gives us the solution for $h(x)$. By the definition of ψ

$$(f+g)(x) = 2\,\frac{\alpha^2\phi(x) - 1}{\lambda}. \tag{3.23}$$

Now, adding the previous equation and (3.21) we get $f(x)$,

$$2f(x) = 2\,\frac{\alpha^2\phi(x) - 1}{\lambda} + \gamma(x)$$
$$f(x) = \frac{\alpha^2\phi(x) - 1}{\lambda} - \frac{\gamma(x)}{2}.$$

Subtracting (3.21) from (3.23) we get $g(x)$,

$$2g(x) = 2\,\frac{\alpha^2\phi(x) - 1}{\lambda} - \gamma(x)$$
$$g(x) = \frac{\alpha^2\phi(x) - 1}{\lambda} - \frac{\gamma(x)}{2},$$

which completes the proof. □

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