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Short note 3 in 1: A simple way to prove that e^r, $\ln(r)$ and π^2 are irrational

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In 1947, apparently inspired by the classical Hermite method to show the transcendence of e (see, e.g., [3, Chapter 2]), Ivan Niven published his famous note "A simple proof that π is irrational" [6]. The key to this proof is the use of sums of different derivatives of special polynomials in order to construct a sequence (P_n) of polynomials with integer coefficients and of degree $\leq n$ fulfilling $0 < |P_n(\pi)| < 1/n!$. If, in this setting, we suppose $\pi = a/b$ to be a rational number, then $b^n P_n(\pi)$ would be an integer number with $0 < |b^n P_n(\pi)| < 1$ for all large n. This is of course impossible and hence π is irrational.

Soon after, Iwamoto [4] and Butlewski [2] exploited variations of Niven's method and constructed other approximation polynomials in order to get simple irrationality proofs for π^2, resp. e^k for any integer $k \neq 0$. In all cases the used polynomials seem to appear from nowhere and to show that they are actually in $\mathbb{Z}[x]$ is more or less tricky.

In this note we take a new look at the classic analytic irrationality proofs for π^2 and the integer powers of e, showing that the required approximation polynomials are generated by one single integral expression. Our approach makes it obvious how the polynomials come into existence, why they have integer coefficients and that the irrationality proofs for π, π^2 and e^k are only different special cases derived from the same general formula.

We start by studying the integral

$$I_k(z) := z^{k+1} \int_0^1 t^k e^{zt}\, dt$$

for $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$. With integration by parts we get the recursive formula

$$I_k(z) = z^k e^z - k I_{k-1}(z) \tag{1}$$

and considering $I_0(z) = e^z - 1$ we see (by induction over k) that for every $k \in \mathbb{N}_0$ there is a polynomial $r_k(z) \in \mathbb{Z}[z]$ of degree k such that

$$I_k(z) = r_k(z) e^z - (-1)^k k!\,.$$

Now, we consider the polynomials

$$p_n(t) := t^n(1-t)^n/n!\,,$$

where n is a positive integer. It is well known (and easily seen) that $p_n^{(n)}(t)$ is a polynomial of degree n with integer coefficients $p_{n,0}, \ldots, p_{n,n}$. Thus the integral

$$J_n(z) := z^{n+1}\int_0^1 p_n^{(n)}(t)e^{zt}\,dt = \sum_{k=0}^{n} p_{n,k}z^{n-k}I_k(z) \tag{2}$$

is the sum of integer multiples of the integrals $z^n I_0(z), z^{n-1}I_1(z), \ldots, I_n(z)$. Hence there are two polynomials $Q_n(z), R_n(z) \in \mathbb{Z}[z]$ of degree $\leq n$ with

$$J_n(z) = Q_n(z) + R_n(z)e^z. \tag{3}$$

On the other hand, by repeatedly using integration by parts we can transform the integral $J_n(z)$ into

$$J_n(z) = (-1)^n\frac{z^{2n+1}}{n!}\int_0^1 t^n(1-t)^n e^{zt}\,dt. \tag{4}$$

Since $0 \leq t(1-t) \leq 1/4$ for all $t \in [0,1]$ we get

$$|J_n(z)| \leq |z|e^{\mathrm{Re}(z)}\frac{|z/2|^{2n}}{n!}. \tag{5}$$

Now suppose that z and e^z are Gaussian rationals, that is, $z \in \mathbb{Q} + \mathbb{Q}\mathbf{i}$ and $e^z \in \mathbb{Q} + \mathbb{Q}\mathbf{i}$. Then $z = (a + a'\mathbf{i})/b$ and $e^z = (c + c'\mathbf{i})/d$ with $a, a', c, c' \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. From (5) we have $db^n J_n(z) \to 0$ as $n \to \infty$ and from (3) we see that

$$db^n J_n(z) = db^n\big(Q_n(z) + R_n(z)e^z\big) \in \mathbb{Z} + \mathbb{Z}\mathbf{i},$$

i.e., a Gaussian integer. From this we obtain the following

Proposition. *If $J_n(z)$ does not eventually vanish then not both of z and e^z can be Gaussian rationals.*

If $x \neq 0$ is a *real* number, then obviously $J_n(x)$ is nonzero, since the integral in (4) is positive. Thus, if $x \neq 0$ is rational, the proposition implies the irrationality of e^x and if $y \neq 1$ is a positive rational number, the proposition (applied to $z = \ln y$) shows the irrationality of $\ln(y)$. For $z = \mathbf{i}\pi$ we have

$$\mathrm{Im}\left(\int_0^1 t^n(1-t)^n e^{zt}\,dt\right) = \int_0^1 t^n(1-t)^n \sin(\pi t)\,dt > 0 \tag{6}$$

and in particular $J_n(\mathbf{i}\pi) \neq 0$. From $e^{\mathbf{i}\pi} = -1$ and the proposition we obtain the irrationality of π. Moreover, since $\cos(s) = -\cos(\pi - s)$, the real part of the integral in (6) vanishes and we obtain from (4) that $J_n(\mathbf{i}\pi)$ is real. Denoting the coefficients of $(Q_n - R_n)(z) \in \mathbb{Z}[z]$ by $c_{n,0}, \ldots, c_{n,n}$ we have

$$0 \neq J_n(\mathbf{i}\pi) = (Q_n - R_n)(\mathbf{i}\pi) = \mathrm{Re}(Q_n - R_n)(\mathbf{i}\pi) = \sum_{\nu=0}^{\lfloor n/2 \rfloor} (-1)^\nu c_{n,2\nu} \pi^{2\nu}.$$

If we suppose that $\pi^2 = a/b$ with positive integers a and b then $b^n J_n(\mathbf{i}\pi)$ is a nonzero integer. Similarly as above, this contradicts (5) for large n.

Remark. (3) and (4) imply that $-Q_n/R_n$ is the (n,n)-Padé approximant of e^z (cf. [1, p. 318]). By explicitly computing the approximation polynomials used by Iwamoto, one then sees that these actually equal $Q_n - R_n$. The same can be observed regarding Niven's polynomials and $\mathrm{Re}(Q_n + R_n\mathbf{i})$ which correspond to the case $z = \mathbf{i}\pi/2$. Also, in the note [5] Nesterenko used the explicit form of the Padé approximants to obtain an estimate for J_n similar to (5) and with that the irrationality of π and e^x for $x \in \mathbb{Q} \setminus \{0\}$.

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