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Calculus style proof of Poncelet's theorem for two ellipses

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1 Introduction

In 1813 Jean-Victor Poncelet (1788–1867) discovered the following celebrated theorem which belongs to classical geometry.

Theorem 1 (Poncelet's closure theorem)**.** *Let $\mathcal{E}_{\mathrm{o}}$ and $\mathcal{E}_{\mathrm{i}}$ be two ellipses with $\mathcal{E}_{\mathrm{i}}$ inside $\mathcal{E}_{\mathrm{o}}$. Suppose that there exists an n-sided polygon circumscribed between $\mathcal{E}_{\mathrm{o}}$ and $\mathcal{E}_{\mathrm{i}}$ that is inscribed in $\mathcal{E}_{\mathrm{o}}$ and circumscribed about $\mathcal{E}_{\mathrm{i}}$. Then for any other point of $\mathcal{E}_{\mathrm{o}}$ there exists an n-sided polygon circumscribed between $\mathcal{E}_{\mathrm{o}}$ and $\mathcal{E}_{\mathrm{i}}$, which has this point for one of its vertices.*

We say that a point of $\mathcal{E}_{\mathrm{o}}$ has the n-Poncelet property if it is one of the vertices of a circumscribed n-sided polygon between $\mathcal{E}_{\mathrm{o}}$ and $\mathcal{E}_{\mathrm{i}}$. Poncelet's closure theorem does not

Der Schliessungssatz von Poncelet gilt als eines der schönsten und tiefsten Resultate der klassischen projektiven Geometrie. Die zahlreichen heute bekannten Beweise sind deutlich schwieriger als etwa der elementare Inversionsbeweis des Schliessungssatzes von Steiner. Poncelet fand seinen Satz während er von Frühling 1813 bis Sommer 1814 in Saratow an der Wolga in russischer Kriegsgefangenschaft sass. Neben seinen fundamentalen Beiträgen zur projektiven Geometrie war Poncelet zu Lebzeiten auch für seine Leistungen als Ingenieur bekannt. So prangt sein Name neben denen von 71 weiteren eminenten Wissenschaftlern am Eiffelturm. Bereits vor Poncelet hatten Chapple, Euler und Fuss Spezialfälle des Schliessungssatzes gefunden. Die Beweismethoden reichen heute von Abelschen Integralen über elliptische Kurven bis zur Masstheorie. Der Autor der vorliegenden Arbeit betrachtet den Spezialfall von zwei Ellipsen, wovon die eine im Inneren der andern liegt. Er verwendet Ideen von Jacobi und Bertrand und konstruiert zum Beweis eine reelle periodische Funktion, bei der schliesslich eine Substitution für ein bestimmtes Integral zum Zuge kommt.

guarantee that such point exists; Cayley's theorem provides a criterion for this in terms of the equations of $\mathcal{E}_\mathrm{o}$ and $\mathcal{E}_\mathrm{i}$, see [Fla09, Chapter 10]. But if it exists, then Theorem 1 is equivalent to the statement that every point of $\mathcal{E}_\mathrm{o}$ has the n-Poncelet property, see Figure 1.

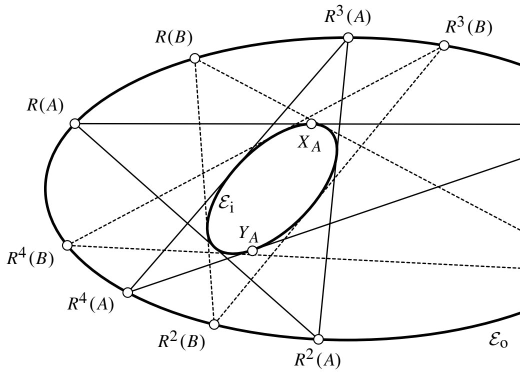

Figure 1 Illustration of Poncelet's theorem. Because the point A has the 5-Poncelet property every point of $\mathcal{E}_\mathrm{o}$ (for instance, B) has the 5-Poncelet property.

Poncelet's theorem also holds for two nondegenerate conics in general position in the projective plane. The reader may consult [Fla09] for a general overview on the subject and its rich history, and [HH15] for the most recent proof.

Theorem 1 is known as the real case of Poncelet's theorem since it considers two ellipses in the real affine plane. Jacobi and, later, Bertrand are credited to have given the first correct proof with the help of elliptic functions. Schoenberg [Sch83] reduced Poncelet's theorem in a non-elementary fashion to the case where $\mathcal{E}_\mathrm{i}$ is a circle having its center in the center of $\mathcal{E}_\mathrm{o}$, and then continued with the Jacobi–Bertrand idea. King in [Kin94] followed Schoenberg's approach, but avoided elliptic integrals and Schoenberg's non-elementary reduction to construct a measure on $\mathcal{E}_\mathrm{o}$ which is invariant with respect to the map $R\colon \mathcal{E}_\mathrm{o} \to \mathcal{E}_\mathrm{o}$; see the beginning of Section 2 for the definition. His proof is reproduced in [Fla09, Chapter 12] together with a section on topological conjugacy between R and a rotation of a circle.

It is fair to say that we make a change of viewpoint, not a change in King's proof, to give a "higher mathematics" or "calculus" style proof of Theorem 1. By this we mean that we construct a periodic continuously differentiable real function $\tilde{R}$ whose derivative is given in terms of tangential distances between the outer and inner ellipse (see Theorem 2). To say that a point with the n-Poncelet property exists is equivalent to $\tilde{R}^n(t) = t + 2k\pi$ being true for some $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, where the superscript means the nth iterated function. In Section 3 we apply this function to a special invariant definite integral through a theorem on a change of variable. This allows us to say that the area under the graph of some positive continuous function between $\tilde{R}^m(t_1)$ and $\tilde{R}^m(t_2)$ is always the same for every $m \in \mathbb{N}$. This is our substitute for the invariant measure.

We hope that this approach will captivate the attention of nonmathematicians as well as advanced high-school students interested in mathematics. Anyway, it is a nice example of simple calculus techniques used to solve a purely geometrical problem.

2 Preparation

Let $\mathcal{E}_\mathrm{o}$ and $\mathcal{E}_\mathrm{i}$ be two ellipses with $\mathcal{E}_\mathrm{i}$ inside $\mathcal{E}_\mathrm{o}$, see Figure 1. Take an arbitrary point $A \in \mathcal{E}_\mathrm{o}$. Then there exist two tangents to $\mathcal{E}_\mathrm{i}$ from A. Choose the one that is on the right-hand side relative to A and denote the other intersection point of this tangent with $\mathcal{E}_\mathrm{o}$ by $R(A)$. This defines the map $R\colon \mathcal{E}_\mathrm{o} \to \mathcal{E}_\mathrm{o}$. Similarly, we can define the map $L\colon \mathcal{E}_\mathrm{o} \to \mathcal{E}_\mathrm{o}$ by choosing the other tangent. Then a point A has the n-Poncelet property if and only if $R^n(A) = A$. Let X_A be the intersection point between the line $R(A)A$ and $\mathcal{E}_\mathrm{i}$, and let Y_A be the intersection point between the line $L(A)A$ and $\mathcal{E}_\mathrm{i}$. Define $\rho_R(A) := |AX_A|$ and $\rho_L(A) := |AY_A|$, where $|\cdot|$ means length of a segment.

Theorem 2. *Let $\mathcal{E}_\mathrm{o}$ and $\mathcal{E}_\mathrm{i}$ be two ellipses with $\mathcal{E}_\mathrm{i}$ inside $\mathcal{E}_\mathrm{o}$. There exist a map $\varphi\colon \mathbb{R} \to \mathcal{E}_\mathrm{o}$ and a function $\tilde{R}\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi\colon [0, 2\pi) \to \mathcal{E}_\mathrm{o}$ is bijective,*

$$R \circ \varphi \equiv \varphi \circ \tilde{R}, \tag{1}$$

$\varphi(t+2k\pi) = \varphi(t)$ and $\tilde{R}(t+2k\pi) = \tilde{R}(t)+2k\pi$ for every $k \in \mathbb{Z}$ and $t \in \mathbb{R}$. Furthermore, $\tilde{R}$ is continuously differentiable and

$$\tilde{R}'(t) = \frac{(\rho_L \circ R \circ \varphi)(t)}{(\rho_R \circ \varphi)(t)}. \tag{2}$$

Proof. Take an orthogonal coordinate system (x, y) with axes parallel to the axes of symmetry of $\mathcal{E}_\mathrm{o}$ and with the origin at the center of this ellipse. Let $\varphi\colon \mathbb{R} \to \mathcal{E}_\mathrm{o}$ be the map $\varphi(t) := (a\cos t, b\sin t)$ where a and b are the semi major and minor axes of $\mathcal{E}_\mathrm{o}$. Then φ restricted to $[0, 2\pi)$ is bijective and has the required periodicity property.

Define the real function $\tilde{R}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{R}(t) := \min\{T : T > t, \varphi(T) = (R \circ \varphi)(t)\}.$$

Then (1) follows immediately from this definition and since φ has the period 2π, $\tilde{R}$ has the desired property. Note also that $\tilde{R}$ is a strictly increasing function, due to the assertion $T > t$.

Define a bijective map $\Lambda\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\Lambda(x, y) := (x/a, y/b)$. This linear transformation maps $\mathcal{E}_\mathrm{o}$ to a circle $\mathcal{K}_\mathrm{o}$ with radius equal to one while $\Lambda(\mathcal{E}_\mathrm{i})$ is also an ellipse inside $\mathcal{K}_\mathrm{o}$. The idea is to work with $\mathcal{K}_\mathrm{o}$ instead of $\mathcal{E}_\mathrm{o}$ by defining maps similar to φ and R, namely $\overline{\varphi}(t) := (\Lambda \circ \varphi)(t) = (\cos t, \sin t)$ and

$$\overline{R}(t) := \left(\overline{\varphi} \circ \tilde{R}\right)(t) = \left(\cos \tilde{R}(t), \sin \tilde{R}(t)\right). \tag{3}$$

It is clear that in the fraction (2) both functions are continuous. Therefore, (2) remains to be proven.

We advise the reader to consult Figure 2. Choose $t \in \mathbb{R}$. Take an arbitrary small $\varepsilon > 0$ and let $|h| < \varepsilon$. With this we assure that $\left|\tilde{R}(t+h) - \tilde{R}(t)\right|$ is small enough. By (3) we have

$$\left|\overline{R}(t+h), \overline{R}(t)\right| = 2\sin\left|\frac{\tilde{R}(t+h) - \tilde{R}(t)}{2}\right|, \quad |\overline{\varphi}(t), \overline{\varphi}(t+h)| = 2\sin\left|\frac{h}{2}\right|.$$

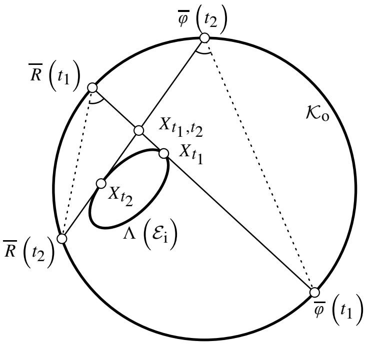

Figure 2 Elements in the proof of Theorem 2.

Let X_{t_1,t_2} be the intersection point between the lines $\overline{\varphi}\,(t_1)\,\overline{R}\,(t_1)$ and $\overline{\varphi}\,(t_2)\,\overline{R}\,(t_2)$. Assume also $h \neq 0$. Since a triangle $\triangle\big(\overline{R}(t+h), X_{t,t+h}, \overline{R}(t)\big)$ is similar to a triangle $\triangle\big(\overline{\varphi}(t), X_{t,t+h}, \overline{\varphi}(t+h)\big)$ by the inscribed angle theorem, it follows

$$\frac{\left|\overline{R}(t+h), \overline{R}(t)\right|}{|\overline{\varphi}(t), \overline{\varphi}(t+h)|} = \frac{\left|\overline{R}(t), X_{t,t+h}\right|}{\left|X_{t,t+h}, \overline{\varphi}(t+h)\right|}.$$

Using the inequalities $\sin x \le x$ and $\sin x \ge x(1 - x^2/6)$ for $x \ge 0$ we get

$$\left|\frac{\tilde{R}(t+h) - \tilde{R}(t)}{h}\right| \ge \left(1 - \frac{h^2}{24}\right) \frac{\left|\overline{R}(t), X_{t,t+h}\right|}{\left|X_{t,t+h}, \overline{\varphi}(t+h)\right|},$$

$$\left|\frac{\tilde{R}(t+h) - \tilde{R}(t)}{h}\right| \le \left(1 - \frac{\left(\tilde{R}(t+h) - \tilde{R}(t)\right)^2}{24}\right)^{-1} \frac{\left|\overline{R}(t), X_{t,t+h}\right|}{\left|X_{t,t+h}, \overline{\varphi}(t+h)\right|}.$$

Since $\tilde{R}$ is an increasing function, we can delete absolute values from the latter inequalities to obtain

$$\tilde{R}'(t) = \lim_{h\to 0} \frac{\tilde{R}(t+h) - \tilde{R}(t)}{h} = \lim_{h\to 0} \frac{\left|\overline{R}(t), X_{t,t+h}\right|}{\left|X_{t,t+h}, \overline{\varphi}(t+h)\right|} = \frac{\left|\overline{R}(t), X_t\right|}{|X_t, \overline{\varphi}(t)|}$$

where X_t is the intersection point between $\overline{\varphi}\,(t)\,\overline{R}\,(t)$ and the ellipse $\Lambda\,(\mathcal{E}_{\mathrm{i}})$, unique by construction of the map R. It is easy to deduce that Λ maps lines to lines and preserves ratios of distances between collinear points. Because $\overline{R}(t)$, X_t and $\overline{\varphi}(t)$ are collinear points, so are $\left(\Lambda^{-1} \circ \overline{R}\right)(t) = (R \circ \varphi)(t)$, $\Lambda^{-1}\,(X_t) = X_{\varphi(t)}$ and $\left(\Lambda^{-1} \circ \overline{\varphi}\right)(t) = \varphi(t)$, and

$$\frac{\left|\overline{R}(t), X_t\right|}{|X_t, \overline{\varphi}(t)|} = \frac{\left|(R\circ\varphi)(t), X_{\varphi(t)}\right|}{\left|X_{\varphi(t)}, \varphi(t)\right|} = \frac{\rho_L\,(R(\varphi(t)))}{\rho_R\,(\varphi(t))}.$$

The proof of Theorem 2 is thus complete. □

Observe that $\varphi(t_1) = \varphi(t_2)$ implies $t_1 = t_2 + 2k\pi$ for some $k \in \mathbb{Z}$. For $n \in \mathbb{N}_0$ we have $R^n \circ \varphi \equiv \varphi \circ \tilde{R}^n$ due to the equation (1). By definition, the point $\varphi(t)$ has the n-Poncelet property if and only if $(R^n \circ \varphi)(t) = \varphi(t)$. Then we deduce the following important fact: *The existence of a point on $\mathcal{E}_\mathrm{o}$ with the n-Poncelet property is equivalent to the existence of $t \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $\tilde{R}^n(t) = t + 2k\pi$. In this case such a point is $\varphi(t)$.*

3 Proof of Poncelet's closure theorem

Here we invoke a theorem on a change of variable in the definite integral: *Assume that $f(t)$ is a continuous function on $[a, b]$, $g(t)$ is a continuously differentiable function on $[A, B]$ and $g([A, B]) = [a, b]$ such that $a = g(A)$ and $b = g(B)$. Then*

$$\int_a^b f(t)\mathrm{d}t = \int_A^B (f \circ g)(t) g'(t)\mathrm{d}t. \tag{4}$$

Take $k \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Additionally, define $\tilde{R}^0(t) = t$. Since $\tilde{R}$ is a strictly increasing continuously differentiable function, it has the required properties for being $g(t)$ in (4). Taking $f(t) = 1/\rho_R(\varphi(t))$ and $A = \tilde{R}^{k-1}(t_1)$, $B = \tilde{R}^{k-1}(t_2)$, we obtain

$$\int_{\tilde{R}^k(t_1)}^{\tilde{R}^k(t_2)} \frac{\mathrm{d}t}{\rho_R(\varphi(t))} = \int_{\tilde{R}^{k-1}(t_1)}^{\tilde{R}^{k-1}(t_2)} \frac{\rho_L(R(\varphi(t)))}{\rho_R(R(\varphi(t)))} \frac{\mathrm{d}t}{\rho_R(\varphi(t))}$$

by using (4) together with (1) and (2). Continuing this process gives

$$\int_{\tilde{R}^k(t_1)}^{\tilde{R}^k(t_2)} \frac{\mathrm{d}t}{\rho_R(\varphi(t))} = \int_{t_1}^{t_2} \frac{\rho_L(R(\varphi(t)))}{\rho_R(R(\varphi(t)))} \cdots \frac{\rho_L\left(R^k(\varphi(t))\right)}{\rho_R\left(R^k(\varphi(t))\right)} \frac{\mathrm{d}t}{\rho_R(\varphi(t))}. \tag{5}$$

Take a linear map Λ which maps $\mathcal{E}_\mathrm{i}$ into a circle $\mathcal{K}_\mathrm{i}$. This map is similar to that in the proof of Theorem 2 except now taking the coordinate system in accordance with the inner ellipse $\mathcal{E}_\mathrm{i}$. Since $\Lambda(\mathcal{E}_\mathrm{o})$ is also an ellipse, it is sufficient to prove Poncelet's closure theorem in cases where the inner ellipse is a circle. But then $\rho_R \equiv \rho_L$ and (5) simplifies to

$$\int_{\tilde{R}^k(t_1)}^{\tilde{R}^k(t_2)} \frac{\mathrm{d}t}{\rho_R(\varphi(t))} = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{\rho_R(\varphi(t))}. \tag{6}$$

This formula justifies the name *invariant integral* and the integrand function is mentioned at the end of the introduction.

In order to prove Theorem 1, let $\varphi(t_1)$ be a point with the n-Poncelet property and take an arbitrary $t_2 > t_1$. We would like to show that $\varphi(t_2)$ has the n-Poncelet property. By (6) we have

$$\begin{aligned}\int_{t_1}^{t_2} \frac{\mathrm{d}t}{\rho_R(\varphi(t))} &= \int_{t_1+2k\pi}^{\tilde{R}^n(t_2)} \frac{\mathrm{d}t}{\rho_R(\varphi(t))} \\ &= \int_{t_1}^{\tilde{R}^n(t_2)-2k\pi} \frac{\mathrm{d}t}{\rho_R(\varphi(t))} = \int_{t_1}^{t_2} + \int_{t_2}^{\tilde{R}^n(t_2)-2k\pi} \frac{\mathrm{d}t}{\rho_R(\varphi(t))}.\end{aligned}$$

The second equality follows due to the fact that $1/\rho_R(\varphi(t))$ is a periodic function with the period 2π. We obtain

$$\int_{t_2}^{\tilde{R}^n(t_2)-2k\pi} \frac{\mathrm{d}t}{\rho_R(\varphi(t))} = 0.$$

Since the integrand is positive, the boundary values of integration must be equal by the mean value theorem for definite integrals. It follows $t_2 = \tilde{R}^n(t_2) - 2k\pi$ and the point $\varphi(t_2)$ also has the n-Poncelet property.

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