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How to find your soulmate: Set your goal vs. aim at dating sites

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1 Introduction

We discuss problems regarding meeting probabilities on undirected graphs with two distinguished vertices, under different planning strategies. We assume that there are maps of the shortest paths between the two vertices and these paths are of equal length. One person is positioned at each vertex and both people select from the stack of maps. They follow the selected paths and move at the same speed. The selection is made uniformly at random and without any prior knowledge of the details of the maps. We call this the goal-oriented approach. The two people may or may not meet en route to their destinations.

In an alternative setup there are two stacks of maps of shortest paths leading from the original starting points only to the set D of potential meeting points, halfway to the original destinations. Again, the two people may or may not meet as they arrive at D. This approach is referred to as the meeting-oriented one. The main goal is to determine the probability of meeting under the two scenarios.

Alice und Bob bewohnen je einen Knoten v_A respektive v_B eines ungerichteten Graphen. Alle endlich vielen kürzesten Wege zwischen v_A und v_B sollen gerade Länge aufweisen. Alice und Bob wählen nun zufällig je einen dieser Wege aus (alle sind gleich wahrscheinlich) und laufen gleichzeitig und gleich schnell von zuhause los. Wie gross ist die Wahrscheinlichkeit, dass sie sich treffen? Ein zweites Szenario: Alice und Bob wählen einen zufälligen kürzesten Weg von zuhause zu einem der möglichen Treffpunkte auf den kürzesten Pfaden aus. Ist die Wahrscheinlichkeit, dass sie sich auf diese Weise treffen grösser, kleiner oder gleich gross wie beim ersten Szenario? Wer die Antwort wissen will, findet sie in der vorliegenden Arbeit.

The motivation comes from the following problem, cf. [3, Problem 3/34] and [5, Problem 929]:

> "Alice and Bob live at opposite corners of the illustrated grid (cf. Figure 1). Each departs for the house of the other at the same time, walking along the grid at the same speed, and choosing one of the many shortest-length paths uniformly at random. What is the probability that they will meet en route? You can assume that Alice and Bob each have a stack of maps. On each map one of the possible shortest routes is highlighted, and the stack consists of all possible such maps. Before leaving, Alice and Bob each choose one of the maps at random, with equal probability, and they follow the indicated route."

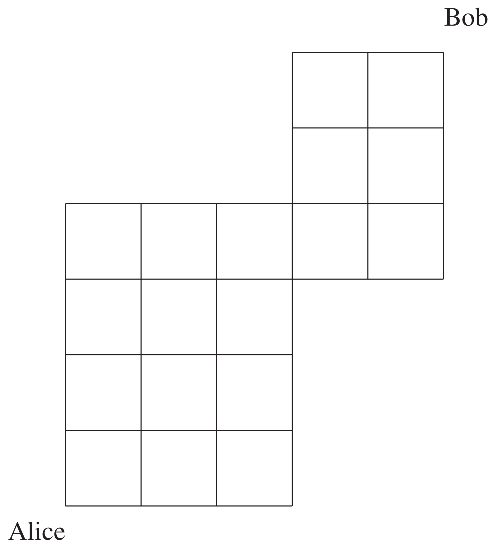

Figure 1 Alice and Bob meet

Our scenario is a bit different. We assume a situation where the shortest paths have even length. This allows two different goals: getting from one place to the other vs. meeting. The former one is the goal-oriented approach with a definite destination while the latter one focuses on "dating" at any of the potential meeting sites. This distinction leads to the notion of full-paths and half-paths that are explained in the following theorem. Note that the selections are made uniformly at random and without any prior knowledge of the details of the maps.

The main goal is to determine the probability of meeting under the two scenarios. We prove that the meeting probability is higher under the goal- (i.e., full-path based) than the meeting-oriented (i.e., half-path based) strategies under some general conditions. We might be also interested in the complementary problem: what if the two people had a fallout and want to avoid each other. Of course, the corresponding probabilities are simply the complements of the meeting probabilities; thus, in this case, somewhat surprisingly, the meeting-oriented approach (i.e., half-path based) gives better probabilities.

We use the following notations and assumptions.

Let $G = (V, E)$ be an undirected graph with two distinguished vertices V_A and V_B. Assume that all shortest paths in G connecting V_A to V_B have the same number l of edges which is an even integer. Two walkers each selects a path uniformly at random: one going from V_A to V_B (Alice) and the other one going from V_B to V_A (Bob). As they walk, they complete exactly one edge during every step; thus, both will need exactly l steps to complete their respective routes.

Let $f(G)$ denote the probability that the two walkers will meet during their travels en route to the opposite endpoint, V_B and V_A, respectively. These routes are called full-paths.

Let D denote the potential meeting sites, i.e., the vertices of G where meeting can take place and let $\mathcal{D} = \{1, 2, \ldots, |D|\}$ denote the index set of the vertices $\{V_1, V_2, \ldots, V_{|D|}\}$ in D. Under another scenario, the two walkers choose paths from V_A to D and V_B to D, respectively, uniformly at random. These paths are called half-paths. Let $h(G)$ denote the probability that the two walkers will meet during their travels, at a vertex in D.

Theorem 1.1. *Let a_i and b_i be the number of paths from V_A to $V_i \in D$ and V_B to $V_i \in D$, respectively. In general, we have the inequality $h(G) \le f(G)$ and equality holds exactly if*

$$a_i = c_1 \quad \text{and} \quad b_i = c_2 \quad \text{for } i \in \mathcal{D} \tag{1}$$

with some positive constants c_1 and c_2 in which case $h(G) = f(G) = 1/|D|$.

The main Theorem 1.1 establishes the fact that in order to maximize meeting probabilities it is better to plan full trips than half trips; thus, claiming the main premise of the introduction. (Although Theorem 1.1 applies to general graphs, the presented examples are all grid graphs.) Theorem 2.2 gives asymptotic results on the underlying probabilities and demonstrates that the asymptotic ratio is $\sqrt{2}$ between the two probabilities for large $n \times n$ symmetric grid graphs.

Remark 1.2. One can be interested in identifying the most likely meeting site. The site or sites in question can be found by maximizing the products $a_i b_i$, $i \in \mathcal{D}$; however, this would assume that the parties are aware of the geography of the graph G.

Our main tool is the use of inequalities for certain power sums. In this paper we assume that $x = (x_1, x_2, \ldots, x_n)$, $x_i \ge 0$. We set

$$M_k^m(x) = \left(\sum_{i=1}^{n} x_i^k \right)^m$$

and $M_k(x) = M_k^1(x)$. Besides the usual inequalities we will use the following one.

Theorem 1.3 ((2.10.1), p. 28, in [2])**.**

$$M_p^a(x) M_r^c(x) \ge M_q^b(x) \tag{2}$$

with $b = a + c$, $ap + cr = bq$, $a, c \ge 0$, and $0 < p < q < r$. The inequality in (2) *is strict unless x is a permutation of $(t, \ldots, t, 0, \ldots, 0)$ with $t \ge 0$.*

We present three special cases in Examples 2.1, 3.1, and 3.2 in Sections 2 and 3. In Section 5, we discuss an alternative strategy planning which might result in even higher meeting probabilities. The main theorem, Theorem 1.1, considers only cases where the number of steps is even. We briefly mention cases when this number is odd in Section 6.

2 Full grids

We start with a simple example where the graph is the $n \times n$ grid.

Example 2.1. Let the graph G_n be the $n \times n$ grid with $n \in \mathbb{N}$. With the notations of Theorem 1.1 we have that $l = 2n$, D consists of the $n+1$ vertices in the NW-SE diagonal, and $a_i = b_i = \binom{n}{i-1}$, $i = 1, 2, \ldots, n+1$ (when listing the vertices of D from NW to SE) by a block walking argument (leading to the Pascal triangle). On Figure 2 there are two labels next to each vertex: the left (right) label shows the number of paths leading to the vertex from the point of view of Alice (Bob).

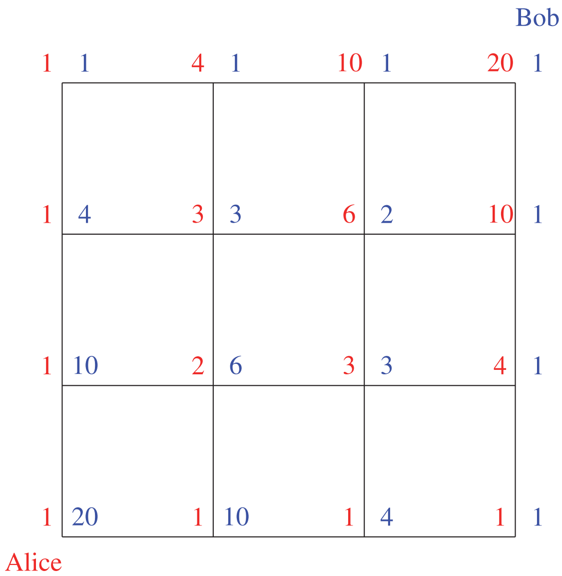

Figure 2 A 3×3 grid

We prove that for grids $G_n, n \geq 2$, mentioned in the above example, we always have the inequality $h(G_n) < f(G_n)$.

Theorem 2.2. *We use the notations of Theorem* 1.1. *Let the graph G_n be the $n \times n$ grid with $n \in \mathbb{N}$. Let $h_n = h(G_n)$ and $f_n = f(G_n)$ be the respective meeting probabilities when picking a half-path and arriving at, or when picking a full-path and passing through vertex $i \in \mathcal{D}$ (en route from V_A to V_B or V_B to V_A). We have*

$$h_n = \sum_{k=0}^{n} \left(\frac{\binom{n}{k}}{2^n} \right)^2$$

and

$$f_n = \sum_{k=0}^{n} \left(\frac{\binom{n}{k}^2}{\binom{2n}{n}} \right)^2 .$$

This yields

$$h_n = \frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}$$

and

$$f_n = \frac{1}{\binom{2n}{n}^2} \sum_{k=0}^{n} \binom{n}{k}^4 \sim \sqrt{\frac{2}{\pi n}}$$

as $n \to \infty$. We have $h_n < f_n$ for $n \geq 2$, and $h_1 = f_1 = f_2 = 1/2$.

3 Partial grids with partial symmetry

We include two examples of partial grids with partial symmetry. We call a symmetry partial if $a_i \neq b_i$ for some $i \in \mathcal{D}$ where a_i and b_i are defined in Theorem 1.1 and of course, a grid is partial if it is not full.

Example 3.1. Figure 3 shows a grid with $f(G) = h(G) = 1/2$.

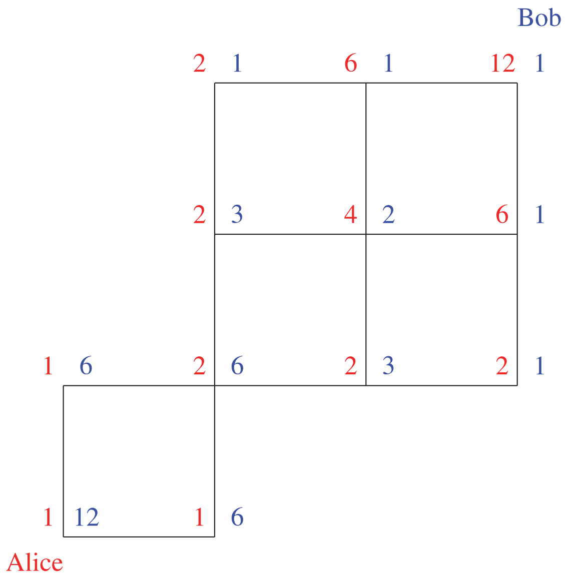

Figure 3 A partial grid with partial symmetry

Example 3.2. Figure 4 shows another partially symmetric grid with $f(G) = 1/2 > h(G) = 12/25$.

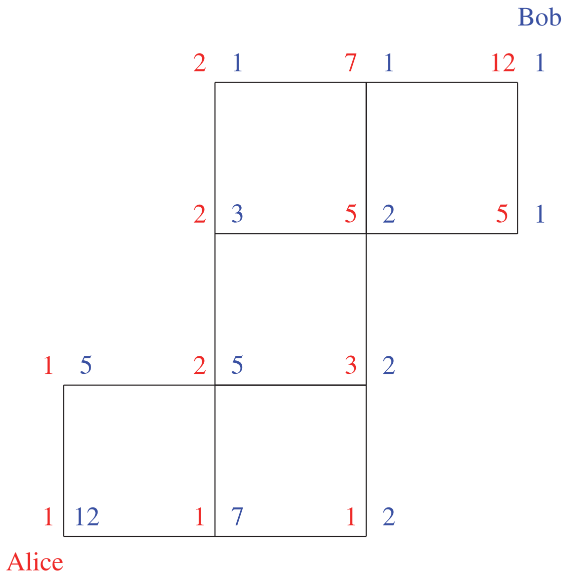

Figure 4 Another partial grid with partial symmetry

4 Proofs

Proof of Theorem 2.2. We easily get the probabilities h_n and f_n, and the inequality $h_n \le f_n$ turns into

$$\binom{2n}{n}^3 \le 2^{2n} \sum_{k=0}^{n} \binom{n}{k}^4.$$

We set $x = (\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n})$. Clearly, $M_1 = M_1(x) = 2^n$ and $M_2 = M_2(x) = \binom{2n}{n}$. We apply (2) of Theorem 1.3

$$M_2^3 \le M_1^2 M_4^1$$

and note that equality applies only if $n = 1$.

The asymptotic results follow by standard calculations, cf. [1, Exercise 38 on p. 90]. For example,

$$f_n = \frac{1}{\binom{2n}{n}^2} \sum_{k=0}^{n} \binom{n}{k}^4 \sim \left(\frac{\sqrt{\pi n}}{2^{2n}}\right)^2 2^{4n-1} \left(\frac{2}{\pi n}\right)^{3/2} = \sqrt{\frac{2}{\pi n}}$$

as $n \to \infty$. □

For the proof of Theorem 1.1 we need some preparation. We introduce the notation $AB = (a_1 b_1, a_2 b_2, \dots, a_n b_n)$, where $A = (a_1, a_2, \dots, a_n)$, and $B = (b_1, b_2, \dots, b_n)$. The Cauchy inequality claims

$$M_1(AB) \le \sqrt{M_2(A) M_2(B)}.$$

The Lagrange identity [4, p. 84] includes the term that turns the alternative form of the Cauchy inequality into an equation:

$$M_1^2(AB) + \frac{1}{2}\sum_{i,j=1}^{n}(a_i b_j - a_j b_i)^2 = M_2(A)M_2(B).$$

It also provides an option for determining the condition under which equality holds in the inequality.

Proof of Theorem 1.1. When using full-paths, the probability that Alice travels through vertex $i \in \mathcal{D}$ en route to V_B is

$$\frac{a_i b_i}{\sum_{j\in\mathcal{D}} a_j b_j};$$

thus, the probability of meeting at $i \in \mathcal{D}$ is

$$\left(\frac{a_i b_i}{\sum_{j\in\mathcal{D}} a_j b_j}\right)^2,$$

and the overall probability of meeting is

$$f(G) = \sum_{i\in\mathcal{D}}\left(\frac{a_i b_i}{\sum_{j\in\mathcal{D}} a_j b_j}\right)^2.$$

In a similar fashion, when both Alice and Bob use half-paths, the meeting probability is

$$h(G) = \sum_{i\in\mathcal{D}}\frac{a_i}{\sum_{j\in\mathcal{D}} a_j}\frac{b_i}{\sum_{j\in\mathcal{D}} b_j}.$$

The relation $h(G) \le f(G)$ is equivalent to

$$M_1^3(AB) \le M_2(AB)M_1(A)M_1(B) \tag{3}$$

with $n = |D|$, which is implied by inequality (2) and the Lagrange identity. In fact, by the Lagrange identity, we get that

$$M_1(A)M_1(B) = M_{1/2}^2(AB) + \frac{1}{2}\sum_{i,j=1}^{|D|}(\sqrt{a_i b_j} - \sqrt{a_j b_i})^2 \ge M_{1/2}^2(AB). \tag{4}$$

On the other hand, inequality (2) results in

$$M_1^3(AB) \le M_2(AB)M_{1/2}^2(AB); \tag{5}$$

thus, (4) implies (3).

Equality applies in (3) if it does in (5) and the second term vanishes in (4). The former one requires that all the terms $a_i b_i$ are equal to some positive number (since the zero terms can be ignored), while the latter one requires that B is some positive multiple of A. In conclusion, it means that $a_i = c_1$ and $b_i = c_2$, $i \in \mathcal{D}$, for some positive constants c_1 and c_2. In this case, every vertex in D will be visited with the same probability (cf. Example 3.1 with $c_1 = 2$, $c_2 = 3$, $|D| = 2$, and $f(G) = h(G) = 1/2$). □

Remark 4.1. If $a_i b_i = d$ for $i \in \mathcal{D}$ then we also have that $f(G) = 1/|D|$ (cf. Examples 3.1 and 3.2 both with $d = 6$, $|D| = 2$, and $f(G) = 1/2$).

5 Different perspectives

What if Alice wants to go from V_A to V_B while Bob only wants to "meet the girl?" This is a somewhat asymmetric situation. It turns out that this is a better plan than simply aiming at D by both Alice and Bob. Let

$$g(G) = \sum_{i \in \mathcal{D}} \frac{a_i b_i}{\sum_{j \in \mathcal{D}} a_j b_j} \frac{b_i}{\sum_{j \in \mathcal{D}} b_j}$$

denote the probability of meeting. We have the inequality $g(G) \geq h(G)$ since

$$M_1(A) M_1(AB^2) \geq M_1^2(AB)$$

with $n = |D|$, by the Cauchy inequality

$$\sum_{i \in \mathcal{D}} \sqrt{a_i}^2 \sum_{i \in \mathcal{D}} \sqrt{a_i b_i^2}^2 \geq \left(\sum_{i \in \mathcal{D}} \sqrt{a_i} \sqrt{a_i b_i^2} \right)^2 .$$

Equality applies exactly if all b_is are equal.

It can also happen, from the point of view of meeting probabilities, that this plan is better than if both of them are goal-oriented, although numerical experimentation suggests that $f(G) > g(G)$ more often happens than the other way around.

Remark 5.1. If $a_i b_i = d$ for $i \in \mathcal{D}$ then $f(G) = g(G) = 1/|D|$.

6 Meeting between two vertices of the grid

The problem mentioned in Section 1 (cf. Figure 1) is represented by the graph $G = (V, E)$ and has $l = 11$ edges in each map. Calculation shows that $f(G) = 0.2913$. In fact, there are four edges where meeting might take place. These edges connect two sets of vertices D_A and D_B. The sets D_A and D_B consist of the vertices that are reached by Alice and Bob, respectively, with index sets $\mathcal{D}_A$ and $\mathcal{D}_B$ of the corresponding vertex sets D_A and D_B. An edge $(V_i, V_j) \in E$ is a potential meeting location if $i \in \mathcal{D}_A$ and $j \in \mathcal{D}_B$. Now let a_i and b_j be the numbers of paths from V_A to $V_i \in D_A$ and V_B to $V_j \in D_B$, respectively. There are $a_i b_j$ paths connecting the vertices V_i and V_j en route from V_A to V_B (and vice versa). We set

$$S = \sum_{\substack{(V_i, V_j) \in E \\ i \in \mathcal{D}_A, j \in \mathcal{D}_B}} a_i b_j$$

and

$$f(G) = \sum_{\substack{(V_i, V_j) \in E \\ i \in \mathcal{D}_A, j \in \mathcal{D}_B}} \left(\frac{a_i b_j}{S} \right)^2 .$$

For the mentioned graph we have $S = 290$ and $f(G) = (30/290)^2 + (60/290)^2 + 2(100/290)^2 = 245/841 = 0.2913$. There is no meaningful definition of half-paths if l is odd.

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