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Lobachevsky-type formulas via Fourier analysis

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Identitäten des *sinus cardinalis* und seiner Potenzen ziehen mindestens seit Shannons berühmtem Abtasttheorem und der in der Signalverarbeitung verwendeten Rekonstruktionsformel periodisch das Interesse von Mathematikern, Physikern und Ingenieuren auf sich. Der vorliegende Artikel beschäftigt sich nun mittels Methoden der Fourier-Analyse mit dem Lobachevski-Integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin(\pi x)}{\pi x}\right)^k p(x)\,dx,$$

wobei p eine gegebene periodische Funktion und k eine positive ganzen Zahl ist. Zur Berechnung des Integrals wird eine Parseval-Formel „gemischten Typs“ verwendet, die eine periodische Funktion f (und ihre Fourier-Koeffizienten) mit einer kompakt getragenen Funktion g (und ihrer Fourier-Transformierten) wie folgt in Beziehung setzt:

$$\int_{\mathbb{R}} f(x)\hat{g}(x)\,dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)g(n).$$

In der Signalverarbeitung entspricht die rechte Seite dieser Gleichung dem Abtasten und Aliasing einer bandbegrenzten Funktion im Frequenzraum.

1 Introduction

Going back to Lobachevsky's original work [6], the following is known as a Lobachevsky-type integral:

$$\int_{-\infty}^{\infty} \left(\frac{\sin(\pi x)}{\pi x} \right)^{k} p(x)\, dx. \tag{1}$$

Here $k \in \mathbb{N} \setminus \{0\}$ is a positive integer, and $p : \mathbb{R} \to \mathbb{C}$ is a periodic function with period $T > 0$ that is assumed to be integrable over a single period. Recently, Jolany [4] has published identities for this integral when k is even and p is a continuous function of period $T = 1$, using methods of complex analysis. We will base our discussion on the Fourier transform and obtain corresponding identities for all $k \in \mathbb{N} \setminus \{0\}$ and p integrable of arbitrary period T.

In the setting of information and communication theory, the sine cardinal generates the Shannon basis, and there exist a plethora of identities that are "folklore" in the signal processing community. For example, the "reconstruction formula" using the Shannon basis in signal processing is just the cardinal series expansion of the mathematical literature [11]. Such identities have inspired the present discussion.

In mathematics, too, there is periodically renewed interest in the surprising properties of integrals involving the sine cardinal, including some recent discussion [2, 3, 7, 12, 13]. It turns out that Fourier transform theory both readily explains these phenomena and provides an interpretation in terms of signal processing.

The "mixed-type" Parseval formula that we develop below yields not only formulas for the Lobachevsky-type integral (1) but can also be used to generate identities for similar integrals involving functions other than the cardinal sine. We give two examples where certain Bessel functions play a role.

2 Parseval formula

For functions $f : \mathbb{R} \to \mathbb{C}$, we define the Fourier transform as follows:

$$\hat{f}(\xi) = (\mathcal{F} f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx, \tag{2}$$

whenever the integral exists. Similarly, the inverse Fourier transform is defined by

$$\check{f}(\xi) = (\mathcal{F}^{-1} f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x \xi}\, dx,$$

again, whenever the integral exists.

The set of absolutely integrable functions on the real axis is denoted by $L^1(\mathbb{R})$, while $\mathrm{BV}(\mathbb{R})$ denotes the set of functions that are of bounded variation on $\mathbb{R}$.

In this treatment, complex-valued, periodic functions of period $T > 0$ that are absolutely integrable over a single period play a major role. The set of such functions is denoted by $L^1([-\frac{T}{2}, \frac{T}{2}])$. In the case $T = 1$, we will write simply $L^1(\mathbb{T})$.

For $f \in L^1([-\frac{T}{2}, \frac{T}{2}])$, we define the Fourier coefficient

$$\hat{f}(n) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2\pi i \frac{nx}{T}} f(x)\, dx, \quad n \in \mathbb{Z}.$$

The minor clash of notation with (2) should not give rise to confusion [8].

We will also use the standard notation

$$f(x^-) := \lim_{\varepsilon \searrow 0} f(x-\varepsilon), \quad f(x^+) := \lim_{\varepsilon \searrow 0} f(x+\varepsilon)$$

for any function f on $\mathbb{R}$ and $x \in \mathbb{R}$.

The classical strong form Parseval formula from Zygmund [14, Theorem 8.18, Chapter IV] of the so-called "mixed type" (i.e., a periodic and a non-periodic function) may be formulated as follows.

Theorem 1 (Parseval formula of mixed type in the strong form). *Let $f \in L^1(\mathbb{T})$ and $g \in L^1(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} \overline{f(x)} g(x)\, dx = \sum_{n=-\infty}^{\infty} \overline{\hat{f}(n)} \hat{g}(n), \tag{3}$$

where $\overline{f(x)}$ denotes the complex conjugate of $f(x)$.

The drawback of this theorem is that the condition on g is frequently difficult to check: it may not be easy to show that g is of bounded variation. More significantly, in certain interesting situations, e.g., where the cardinal sine is involved, Theorem 1 is simply not applicable.

Therefore, we establish the following result in the spirit of Titchmarsh [10, Theorem 47], which yields a "weak-form" Parseval formula of mixed type under the condition that the Fourier transform of g, rather than g itself, is of compact support and of bounded variation at appropriate points. We denote the support of a function g by $\operatorname{supp} g$.

Theorem 2 (Parseval formula of mixed type in the weak form). *Let $f \in L^1(\mathbb{T})$ and $g \in L^1(\mathbb{R})$, and suppose that there exists some $A > 0$ such that $\operatorname{supp} g \subset [-A, A]$. Further, let g be of bounded variation in neighborhoods of all $n \in \mathbb{Z}$ with $|n| \le A$. Then*

$$\int_{-\infty}^{\infty} f(x) \hat{g}(x)\, dx = \sum_{\substack{n \in \mathbb{Z} \\ |n| \le A}} \hat{f}(n) \cdot \frac{g(n^-) + g(n^+)}{2}.$$

Theorem 2 can be adapted to periodic functions p with arbitrary period $T > 0$ by setting $f(x) := p(Tx)$, yielding the following.

Corollary 3. *Let $p \in L^1([-\frac{T}{2}, \frac{T}{2}])$, $g \in L^1(\mathbb{R})$, and suppose that there exists some $A > 0$ such that $\operatorname{supp} g \subset [-A, A]$. Further, let g be of bounded variation in neighborhoods of all points $\frac{n}{T}$, $n \in \mathbb{Z}$, with $|\frac{n}{T}| \le A$. Then*

$$\int_{-\infty}^{\infty} p(x) \hat{g}(x)\, dx = \sum_{\substack{n \in \mathbb{Z} \\ |\frac{n}{T}| \le A}} \hat{p}(n) \cdot \frac{g((\frac{n}{T})^-) + g((\frac{n}{T})^+)}{2}. \tag{4}$$

Functions of compact support are related to signals that are "band-limited" in the parlance of the signal processing community, since it is possible to recover a band-limited (continuous) signal by appropriate (discrete) sampling [11].

Example 1. Consider $\psi_1 \in L^1(\mathbb{R})$ given by

$$\psi_1(x) = \begin{cases} \sqrt{1-x^2}, & -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Its Fourier transform is given by

$$\hat{\psi}_1(\xi) = \begin{cases} \dfrac{J_1(2\pi\xi)}{2\xi}, & x \neq 0, \\ \dfrac{\pi}{2}, & x = 0, \end{cases}$$

where J_1 is the Bessel function of the first kind of order one. (The function given by $\frac{J_1(\xi)}{\xi}$ and its scaled versions are sometimes called Sombrero function, besinc function, or jinc function.) A Parseval formula (either (4) or (3)) then gives

$$\int_{-\infty}^{\infty} \frac{J_1(2\pi x)}{2x} p(x)\, dx = \sum_{|\frac{n}{T}|<1} \hat{p}(n) \sqrt{1-\left(\frac{n}{T}\right)^2}$$

for any $p \in L^1([-\frac{T}{2}, \frac{T}{2}])$. If $0 < T \le 1$, only the summand for index $n = 0$ remains, and we have

$$\int_{-\infty}^{\infty} \frac{J_1(2\pi x)}{2x} p(x)\, dx = \hat{p}(0) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} p(x)\, dx.$$

To apply Corollary 3, we need to check that ψ_1 is of bounded variation at least locally near any point of $\mathbb{R}$, which is not difficult. On the other hand, invoking Theorem 1 would entail verifying that $\hat{\psi}_1$ is of bounded variation, a much more difficult task.

Example 2. Now consider $\psi_2 \in L^1(\mathbb{R})$ given by

$$\psi_2(x) = \begin{cases} \dfrac{1}{\sqrt{1-x^2}}, & -1 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

with Fourier transform

$$\hat{\psi}_2(\xi) = \pi J_0(2\pi\xi),$$

where J_0 is the Bessel function of the first kind of order zero. Since $J_0 \notin L^1(\mathbb{R})$, Theorem 1 cannot be applied.

Observe that $\operatorname{supp} \psi_2 = [-1, 1]$ and that ψ_2 is of bounded variation in the neighborhoods of any point except ± 1. Then, for $T \notin \mathbb{N}$, we can apply Corollary 3 to $p \in L^1([-\frac{T}{2}, \frac{T}{2}])$ and obtain

$$\pi \int_{-\infty}^{\infty} J_0(2\pi x) p(x)\, dx = \sum_{|\frac{n}{T}|<1} \frac{\hat{p}(n)}{\sqrt{1-(\frac{n}{T})^2}}.$$

As before, if $0 < T < 1$, we have

$$\pi \int_{-\infty}^{\infty} J_0(2\pi x) p(x)\, dx = \hat{p}(0) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} p(x)\, dx.$$

3 Lobachevsky integral formulas

We define the usual convolution of $f, g \in L^1(\mathbb{R})$ by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(y) g(x-y)\, dy$$

and write

$$f^{*k} := \underbrace{f * f * \cdots * f}_{k \text{ times}}.$$

We introduce the real function Π given by

$$\Pi(x) := \begin{cases} 1, & |x| < \frac{1}{2}, \\ \frac{1}{2}, & x = \pm\frac{1}{2}, \\ 0, & |x| > \frac{1}{2}, \end{cases}$$

as well as the normalized cardinal sine on $\mathbb{R}$,

$$\operatorname{sinc}(x) := \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

We remark that $\operatorname{sinc} = \mathcal{F}\Pi$, and, more generally, $\operatorname{sinc}^k = \mathcal{F}(\Pi^{*k})$ for $k \in \mathbb{N} \setminus \{0\}$. Furthermore, note that $\operatorname{supp} \Pi^{*k} = [-\frac{k}{2}, \frac{k}{2}]$, and Π^{*k} is piecewise polynomial, hence also in $L^1(\mathbb{R})$ and is of bounded variation in neighborhoods of all points in $\operatorname{supp} \Pi^{*k}$. The functions Π^{*k} are also known as centered B-splines (or Lobachevsky splines) in the signal processing community [1, 5, 11].

As a special case of Corollary 3, we have the following theorem on the Lobachevsky integral formula.

Theorem 4. *Let $p \in L^1([-\frac{T}{2}, \frac{T}{2}])$ for some $T > 0$ and $k \in \mathbb{N} \setminus \{0\}$. Then*

$$\int_{-\infty}^{\infty} \operatorname{sinc}^k(x) p(x)\, dx = \sum_{|\frac{n}{T}| \leq \frac{k}{2}} \hat{p}(n) \Pi^{*k}\left(\frac{n}{T}\right).$$

If $k \geq 2$, the range of summation may be reduced as follows:

$$\int_{-\infty}^{\infty} \operatorname{sinc}^k(x) p(x)\, dx = \sum_{|\frac{n}{T}| < \frac{k}{2}} \hat{p}(n) \Pi^{*k}\left(\frac{n}{T}\right).$$

Corollary 5. *Let $k \in \mathbb{N} \setminus \{0\}$ and $p \in L^1([-\frac{T}{2}, \frac{T}{2}])$ for some $T > 0$ with $kT \leq 2$ if $k \geq 2$, or $0 < T < 2$ if $k = 1$. Then*

$$\int_{-\infty}^{\infty} \operatorname{sinc}^k(x) p(x)\, dx = \Pi^{*k}(0) \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} p(x)\, dx.$$

A few identities are then immediate, most notably for $k = 1$ and $k = 2$,

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x) f(x)\, dx = \int_{-\infty}^{\infty} \operatorname{sinc}^2(x) f(x)\, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\, dx \tag{5}$$

for $f \in L^1(\mathbb{T})$. It is not a coincidence that the two cardinal sine integrals yield the same value; this follows from the fact that $\Pi(0) = \Pi^{*2}(0) = 1$ and both Π and Π^{*2} are continuous at zero.

The identities in (5) further reduce to the well-known Dirichlet and Fejér integrals [8] when $f \equiv 1$,

$$\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\pi x}\, dx = \int_{-\infty}^{\infty} \left(\frac{\sin(\pi x)}{\pi x}\right)^2 dx = 1.$$

More interestingly, when $k = 3$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sinc}^3(x) f(x)\, dx &= \hat{f}(0)\Pi^{*3}(0) + \hat{f}(-1)\Pi^{*3}(-1) + \hat{f}(1)\Pi^{*3}(1) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\, dx - \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) \sin^2(\pi x)\, dx \end{aligned} \tag{6}$$

and, when $k = 4$,

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sinc}^4(x) f(x)\, dx &= \hat{f}(0)\Pi^{*4}(0) + \hat{f}(-1)\Pi^{*4}(-1) + \hat{f}(1)\Pi^{*4}(1) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\, dx - \frac{2}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) \sin^2(\pi x)\, dx. \end{aligned} \tag{7}$$

The identity in (7) was previously derived in [4] using complex analytic methods, while (6) is new, as the results in [4] did not extend to odd-valued integers k.

4 The Poisson summation formula

One of the crucial ingredients in the proof of Theorem 2 is a version of the Poisson summation formula [14, equation (13.4)]. It later appeared explicitly in [3, Proposition 1] for functions of compact support. We present the theorem here, along with a clearer proof, which follows Zygmund [14, p. 68].

Theorem 6 (Poisson summation formula). *Let* $g \in L^1(\mathbb{R})$*, and suppose that there exists some* $A > 0$ *such that* $\operatorname{supp} g \subset [-A, A]$*. Further, let* g *be of bounded variation in neighborhoods of all* $n \in \mathbb{Z}$ *with* $|n| \leq A$*. Then*

$$\sum_{m \in \mathbb{Z}} \hat{g}(m) = \sum_{\substack{n \in \mathbb{Z} \\ |n| \leq A}} \frac{g(n^+) + g(n^-)}{2}. \tag{8}$$

The following identity is immediate since g is of compact support:

$$\sum_{m=-\infty}^{\infty} \hat{g}(m + \xi) = \sum_{\substack{n \in \mathbb{Z} \\ |n| \leq A}} \frac{g(n^+) + g(n^-)}{2} e^{-2\pi i n \xi}. \tag{9}$$

Proof of Theorem 6. Following Zygmund [14, p. 68], we define a periodic function G on $\mathbb{R}$ as follows:

$$G(x) := \sum_{k=-\infty}^{\infty} g(x+k).$$

Since $\operatorname{supp} g = [-A, A]$, the sum on the right will be finite for any fixed $x \in \mathbb{R}$. Furthermore, this will be the case also when x varies in $[-\frac{1}{2}, \frac{1}{2}]$, so we can write, for suitable $K \in \mathbb{N}$,

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} |G(x)|\,dx \leq \sum_{k=-K}^{K} \int_{-\frac{1}{2}}^{\frac{1}{2}} |g(x+k)|\,dx &= \sum_{k=-K}^{K} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} |g(x)|\,dx \\ &= \int_{-\infty}^{\infty} |g(x)|\,dx, \end{aligned}$$

where we have again used the boundedness of the support of g. The last integral is finite since $g \in L^1(\mathbb{R})$, and we conclude that $G \in L^1(\mathbb{T})$.

Since g is of bounded variation in neighborhoods of those $n \in \mathbb{Z}$ with $|n| \leq A$, we deduce that G is of bounded variation in a neighborhood of $x = 0$. Therefore, we can apply the Dirichlet–Jordan test for Fourier series [9, p. 406] to deduce that the Fourier series expansion of G converges in a neighborhood of $x = 0$ as follows:

$$\frac{G(x^+) + G(x^-)}{2} = \sum_{m=-\infty}^{\infty} \hat{G}(m) e^{2\pi i m x}. \tag{10}$$

A direct calculation yields

$$\begin{aligned} \hat{G}(m) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} G(x) e^{-2\pi i m x}\,dx \\ &= \sum_{k=-N}^{N} \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x+k) e^{-2\pi i m x}\,dx \\ &= \int_{-\infty}^{\infty} g(x) e^{-2\pi i m x}\,dx = \hat{g}(m). \end{aligned}$$

Setting $x = 0$ in (10) then establishes (8). ∎

The above version of the Poisson summation formula may be applied to functions that are neither of Schwartz class nor of moderate decay (both need to be continuous as required, e.g., in [8]).

For example, neither Π nor sinc are of Schwartz class or moderate decay, yet we can still obtain a meaningful Poisson summation formula for Π by applying Theorem 6. In particular, using $\mathcal{F}[\Pi(\pi(\cdot))](\xi) = \frac{1}{\pi}\operatorname{sinc}(\frac{\xi}{\pi})$, we obtain

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}}\frac{\sin(n)}{n} = \sum_{m\in\mathbb{Z}}\Pi(\pi m) = \Pi(0) = 1.$$

We are now ready to prove Theorem 2.

Proof of Theorem 2. Since f is periodic with period 1, we can write

$$\begin{aligned}
\int_{-\infty}^{\infty}\hat{g}(\xi)f(\xi)\,d\xi &= \lim_{M\to\infty}\sum_{m=-M}^{M}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}}\hat{g}(\xi)f(\xi)\,d\xi \\
&= \lim_{M\to\infty}\sum_{m=-M}^{M}\int_{-\frac{1}{2}}^{\frac{1}{2}}\hat{g}(\xi+m)f(\xi+m)\,d\xi \\
&= \lim_{M\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{m=-M}^{M}\hat{g}(\xi+m)f(\xi)\,d\xi.
\end{aligned}$$

The Poisson summation formula (9) guarantees that the series converges and is bounded, so that by the dominated convergence theorem the limit and the integral can be exchanged. Moreover,

$$\begin{aligned}
\int_{-\infty}^{\infty}\hat{g}(\xi)f(\xi)\,d\xi &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\lim_{M\to\infty}\sum_{m=-M}^{M}\hat{g}(\xi+m)f(\xi)\,d\xi \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{\substack{n\in\mathbb{Z}\\|n|\le A}}\frac{g(n^-)+g(n^+)}{2}e^{-2\pi i n\xi}f(\xi)\,d\xi \\
&= \sum_{\substack{n\in\mathbb{Z}\\|n|\le A}}\frac{g(n^-)+g(n^+)}{2}\hat{f}(n),
\end{aligned}$$

completing the proof. ■

References

[1] G. Allasia, Remarkable Haar spaces of multivariate piecewise polynomials, *Numer. Algorithms* **78** (2018), no. 2, 661–672.

[2] J. Baez, Patterns that eventually fail, Azimuth [Blog], September 2018, https://johncarlosbaez.wordpress.com/2018/09/20/patterns-that-eventually-fail/.

[3] R. Baillie, D. Borwein and J. M. Borwein, Surprising sinc sums and integrals, *Amer. Math. Monthly* **115** (2008), no. 10, 888–901.

[4] H. Jolany, An extension of the Lobachevsky formula, *Elem. Math.* **73** (2018), no. 3, 89–94.

[5] J. A. C. Kolk, On Euler numbers, Hilbert sums, Lobachevskii integrals, and their asymptotics, *Indag. Math. (N.S.)* **14** (2003), no. 3–4, 445–449.

[6] N. I. Lobatschewsky, Probabilité des résultats moyens tirés d'observations répetées, *J. reine angew. Math.* **26** (1842), 164–170.

[7] S. N. Majumdar and E. Trizac, When random walkers help solving intriguing integrals, *Phys. Rev. Lett.* **123** (2019), Article ID 020201.

[8] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton Lect. Anal. 1, Princeton University Press, Princeton, 2011.

[9] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, Oxford, 1939.

[10] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1948.

[11] M. Unser, Sampling—50 years after Shannon, *Proc. IEEE* **88** (2000), no. 4, 569–587.

[12] User "T. Amdeberhan", "sinc-ing" integral, MathOverflow [Online Forum], March 2017, https://mathoverflow.net/q/263482.

[13] User "Lucia", More on "sinc-ing" integrals and sums, MathOverflow [Online Forum], March 2017, https://mathoverflow.net/q/263615.

[14] A. Zygmund, *Trigonometric Series. Volume 1*, Cambridge University Press, Cambridge, 2002.

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