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The smallest convex k-gon containing n congruent disks

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1 Introduction

Efficiently packing a number of objects is one the most frequently encountered practical problems. Since ancient times, it appeared in every household, in small shops and large factories, in puzzles and recreations and probably in the nature too. The simplest non-trivial objects to be packed are congruent spheres and some of the well-known math-

Das Problem, Kugeln oder Kreise möglichst eng zu packen, kann auf ganz unterschiedliche Weise formuliert werden. Die bekannteste Version ist wohl das Keplersche Problem, die dichteste Packung gleich grosser Kugeln im dreidimensionalen Raum zu finden. Endliche Varianten des zweidimensionalen Problems verlangen, eine endliche Anzahl von Kreisen so anzuordnen, dass sie in einem konvexen Container mit minimaler Fläche oder minimalem Umfang Platz finden. Der Container kann beispielsweise die konvexe Hülle sein oder ein konvexes Polygon mit gegebener Eckenzahl. Derartige Probleme gehören zu den Klassikern der Computational Geometry und des Operations Research. Zahlreiche numerische Lösungsverfahren wurden vorgeschlagen, wobei nicht nur Kreise gepackt werden, sondern auch Ellipsen oder konvexe und nicht-konvexe Polygone. Die Autoren der vorliegenden Arbeit betrachten das Problem, ein flächenminimales konvexes Polygon mit einer festen Eckenzahl zu finden, das eine gegebene Anzahl von nicht überlappenden Einheitskreisscheiben enthält. Es werden untere Schranken für die minimale Fläche mithilfe von einfachen geometrischen Konstruktionen angegeben. Es wird auch diskutiert, in welchen Fällen die Schranken scharf sind.

ematical problems on sphere packing are the Kepler conjecture and the kissing number problem [20]. These problems attract our mind by their innocent statements but defy our attempts by their hidden complexities. At the moment, the problem of filling the entire space with congruent spheres as efficiently as possible is solved for 2, 8, 24 dimensions, and the proposed solution of J. Kepler for 3 dimensions is verified [23]. Mathematicians expect that some deep and unexpected connections will be discovered when these problems are solved.

On the other hand, we are free to restrict our attention to finitely many spheres, and this is the approach we follow now. Most of such problems fall into two types [12].

- *Free packing*: Locate a finite set of congruent spheres so that the volume of their convex hull is minimal. In the two-dimensional space, notable results on this problem are the Thue–Groemer and Oler inequalities (see [4, Chapter 4.3], [10]) and the Wegner inequality (see Theorem 2). In higher dimensions, L. F. Tóth's sausage conjecture is a partially solved major open problem [3].
- *Bin packing*: Locate a finite set of congruent spheres in the smallest volume container of a specific kind. In the two-dimensional space, the container is usually a circle [9], an equilateral triangle [15] or a square [16]. In such cases, the smallest containers and the corresponding optimal packings are known when the number of disks is not so big, e.g. up to 20 (see [11]).

The problem that we study in this paper contains elements of both types.

Problem 1. *For $k \geq 3$, find the smallest-area convex k-gon containing $n \in \mathbb{N}$ unit radius ("unit" in brief) disks without an overlap.*

The solution to this problem for $n = 1$ is given by the following well-known result (see [1, Chapter 2]).

Theorem 1. *When $n = 1$, the regular k-gon circumscribing a unit disk is the only solution to Problem 1 for $k \geq 3$.*

In Theorem 4 of this paper, we give an extension of this result as an inequality bounding area of the containing polygon from below. This inequality is tight in many cases including

- $n = 1$ and $k \geq 3$,
- $n = 2$ and $k = 2k'$ for $k' \geq 2$,
- $n \in \{3, 6\}$ and $k = 3k'$ and
- n is a centred hexagonal number and $k = 6k'$.

The solution of Problem 1 for these tight cases is obtained in Theorem 5. Then we discuss its solution for cases where this bound is not tight, i.e. for $n = 2$, $k = 2k' + 1$ in the remark following Theorem 5, and for $n = 3$, $k = 4$ in Theorem 6. The latter case is essential as it demonstrates the possibility of disks being packed non-efficiently inside a minimal polygon. Along the way to prove our main results, we prove two intermediate results which are interesting on their own. The first one gives geometric invariants between two polygons whose sides are pairwise parallel; see Proposition 1. The second one gives a simple geometric characterization for a well-known curve, the trisectrix of Maclaurin; see Proposition 2.

Instead of spheres, one can pack more complicated objects and let us give a glimpse on the research in this direction. In [22], the problem of free packing of two convex polygons is analysed. The authors of [22] parametrize the set of possible contact configurations of two polygons by two parameters, one indicating location and the other indicating orientation or alignment. This idea is reminiscent of the well-known solution of Buffon's needle problem in probability theory. Then, with simple but elegant steps, they reduce the search space for optimal packing into a finite set and provide a fast algorithm to find a solution. Besides establishing correctness, computational complexity of the proposed algorithm is also analysed.

In a series of papers, e.g. [2, 7], the so-called phi-function method was developed and used to solve various packing and cutting problems. According to [7], roots of this method date back to the 1980s, and the main objective is to provide a general setting where such problems can be formulated so that computationally efficient solutions are possible. The phi-function method is a smart combination of ideas in combinatorial topology, discrete geometry (in line with Hilbert's third problem), analytical geometry and optimization. In [2], it is applied to pack two irregular objects, which can have arbitrary strange shapes as long as they satisfy certain topological regularities, into a container (i.e. a circle, rectangle or regular polygon) to minimize area, perimeter or homothetic coefficient. There is also a possibility of imposing additional distance constraints between objects, and objects and the container.

As usual with packing problems, we believe that Problem 1 is computationally hard. The main difficulty comes from two optimization problems embedded in it, namely simultaneously choosing the right configuration of n spheres and finding the right k-gon. The reader is expected to feel this difficulty towards the end of Section 3 and invited to face it in Section 4.

2 Preliminaries

In addition to the usual ones, we use the following definitions. A *region* is a subset of the plane with finite area, and when X is a region, $\|X\|$ denotes its area. The line segment connecting points A, B is denoted as AB; $|AB|$ is its length, and (AB) is its interior. Let a finite set of unit disks be located in $\mathbb{R}^2$ without an overlap, i.e. each pair has a disjoint interior. Their *joint tangent* is a line which is tangent to at least two of the disks and supports their convex hull. Each joint tangent bounds a half plane which contains the disks. The intersection of these half planes is called *tangent polygon* of the disks (Figure 1 (a), (b)). Clearly, every tangent polygon is convex, and it is a convex polygon as long as the centres of the disks are not all collinear.

The regular k-gon circumscribing a unit disk is called *unit k-gon*. Thus, a unit triangle has sides of length $2\sqrt{3}$, while a unit square has sides of length 2. The *$\frac{1}{m}$-th of a regular mk-gon* is a polygon obtained after cutting the original polygon by two apothems intersecting at $\frac{2\pi}{m}$ angle; see Figure 2.

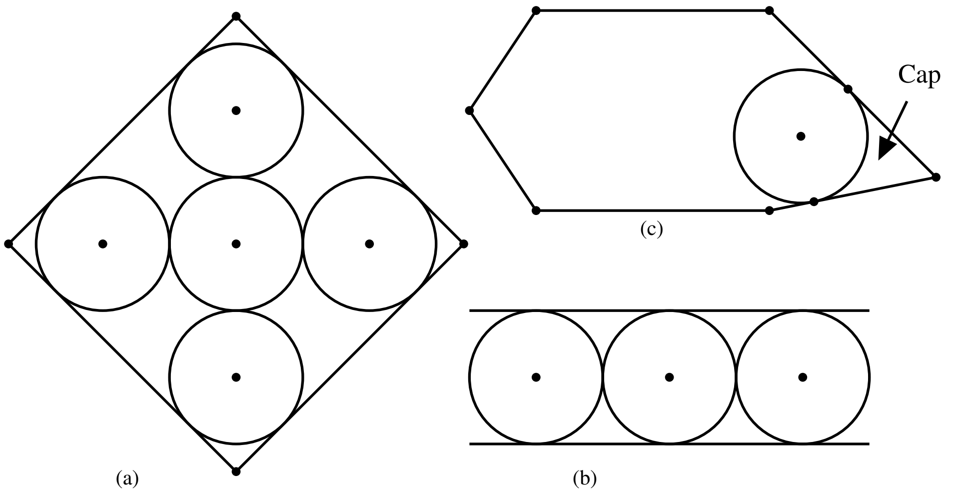


Figure 1. Tangent polygons (a quadrilateral and an infinite strip) and a cap.

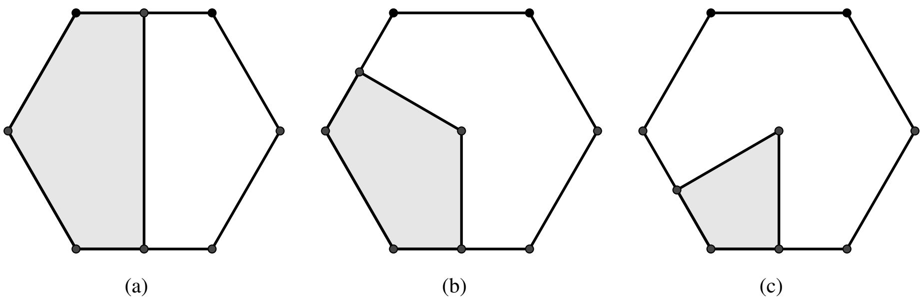


Figure 2. Half, one-third, one-sixth of a regular hexagon.

Let $C \subset \mathbb{R}^2$ be a convex disk, and let P be a convex polygon containing C. Using a terminology of [19], a *cap of P with respect to C* is the region enclosed by the boundary of C and two consecutive sides of P which are tangent to C; see Figure 1 (c). The sum of areas of all caps of P with respect to C is denoted as $\|\mathrm{Cap}_C(P)\|$.

A set of n unit disks constitute a *Groemer packing* if each pair has disjoint interior and the convex hull of their centres is either a line segment of length $2(n-1)$ or can be triangulated into equilateral triangles of edge length two using the n centres as vertices [13]. If, in addition, perimeter of the hull is $2\lceil\sqrt{12n-3}-3\rceil$, where $\lceil x\rceil = \min\{z \in \mathbb{Z} : z \geq x\}$, then the Groemer packing is a *Wegner packing* [5]; see Figure 3. By using these geometric properties, it is easy to show that the convex hull of the centres of the disks in a Wegner packing has at most six sides; see [4, Chapter 4.3]. The number $n \in \mathbb{N}$ is called *exceptional* if there is no Wegner packing of n unit disks. The smallest exceptional number is 121, and they constitute less than 5 % of all $\mathbb{N}$ (see [5]). The following result is known as Wegner inequality; see [4, Chapter 4.3], [5].

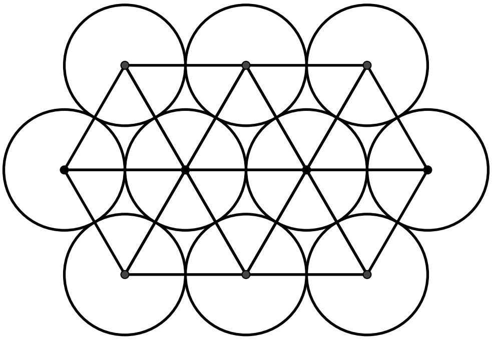


Figure 3. Wegner packing of $n = 10$ disks.

Theorem 2 (Wegner inequality). *If H is the convex hull of $n \in \mathbb{N}$ non-overlapping unit disks, then*

$$\|H\| \geq \sqrt{12} \cdot (n-1) + (2-\sqrt{3}) \cdot \lceil \sqrt{12n-3} - 3 \rceil + \pi.$$

Equality holds if and only if the disks are packed in a Wegner packing.

The following result is a reliable tool when one studies the smallest circumscribing polygons of convex figures [6, 25].

Theorem 3. *Let $C \subset \mathbb{R}^2$ be a convex disk, and let P be the smallest-area convex polygon containing it. Then midpoints of sides of P lie on the boundary of C.*

If an internal angle at a vertex of a polygon is greater than π, then the vertex is *reflex*. If $P = A_1 \dots A_n$ and $Q = B_1 \dots B_n$ are two simple polygons with the same orientation and $A_i A_{i+1} \parallel B_i B_{i+1}$ for all $1 \leq i \leq n$, then they are called *parallel polygons*. Finally, the *Maclaurin trisectrix* is a cubic plane curve defined as the locus of the point of intersection of two lines, each rotating uniformly about a separate point with a ratio of speeds of $\frac{1}{3}$ and the lines initially coincide. Its polar equation is $r = a \sec \frac{\theta}{3}$, and its Cartesian equation is $y^2 = \frac{x^2(x+3a)}{a-x}$ (see [24]).

3 The main results

We shall prove two intermediate results.

Proposition 1. *Let P and Q be two simple parallel polygons.*

(a) *Then they have the same number of reflex vertices.*

(b) *If one is convex, so is the other, and their corresponding internal angles are equal.*

Proof. Let $P = A_1 \dots A_n$ and $Q = B_1 \dots B_n$, and we denote their internal angles as $\measuredangle A_i = \alpha_i$, $\measuredangle B_i = \beta_i$ for $1 \leq i \leq n$. Let $I = \{i \in \mathbb{N} : 1 \leq i \leq n,\ \alpha_i = \beta_i\}$, $I_1 = \{i \in \mathbb{N} : 1 \leq i \leq n,\ \alpha_i < \beta_i\}$ and $I_2 = \{i \in \mathbb{N} : 1 \leq i \leq n,\ \alpha_i > \beta_i\}$. Since $A_i A_{i+1} \parallel B_i B_{i+1}$ for $1 \leq i \leq n$, $i \in I_1$ implies $\alpha_i + \pi = \beta_i$, and $i \in I_2$ implies $\alpha_i = \beta_i + \pi$.

Note that

$$\sum_{i\in I}\alpha_i+\sum_{i\in I_1}\alpha_i+\sum_{i\in I_2}\alpha_i=\sum_{i\in I}\beta_i+\sum_{i\in I_1}\beta_i+\sum_{i\in I_2}\beta_i=(n-2)\pi.$$

Since $\sum_{i\in I}\alpha_i=\sum_{i\in I}\beta_i$, $\sum_{i\in I_1}(\alpha_i+\pi)=\sum_{i\in I_1}\beta_i$ and $\sum_{i\in I_2}\alpha_i=\sum_{i\in I_2}(\beta_i+\pi)$, we get

$$\sum_{i\in I_1}\alpha_i+\sum_{i\in I_2}\beta_i+|I_2|\pi=\sum_{i\in I_1}\alpha_i+|I_1|\pi+\sum_{i\in I_2}\beta_i,$$

which simplifies to $|I_1|=|I_2|$. This proves Proposition 1 (a) after noting that $|I_1|$ is the number of reflex vertices in Q whose corresponding vertex in P is normal, i.e. non-reflex, and $|I_2|$ is the number of reflex vertices in P whose corresponding vertex in Q is normal.

Assume P is convex. Then, by part (a), both polygons have 0 reflex vertices. Hence, Q is convex. Since $n=|I|+|I_1|+|I_2|$ and $|I_1|=|I_2|=0$, we have $n=|I|$, i.e. $\alpha_i=\beta_i$ for all $1\le i\le n$. ■

Remarks. There seems to be a slight confusion in the computational geometry literature regarding to geometric invariants between parallel polygons. For example, on [14, p. 2], it is (mistakenly) claimed that "two polygons are parallel if and only if they have the same sequence of angles". The above result clarifies the situation.

Let P be a convex k-gon containing n unit disks without an overlap such that each side of P is tangent to at least one of the disks. Let us pick one of the disks, and for each side of P, there are two tangents to the disk which are parallel to it. Choose the one which is closer to the side, and the polygon whose sides are contained in these tangents is called *shrink of* P for the picked disk; see Figure 5. By construction, P and its shrink are parallel, and by Proposition 1 (b), they have the same internal angles.

Our second intermediate result is as follows.

Proposition 2. *Let ω be a circle with centre O and radius r_ω, let l be a line tangent to ω at T, and let X be an arbitrary point on l. Let m be the other tangent from X to ω, $R=m\cap\omega$, and let X' be the reflection of X on m with respect to R. Then $t(\omega,l)$, the locus of X', is a Maclaurin trisectrix. Conversely, if $t(\omega,l)$ is a Maclaurin trisectrix, then there exist a circle ω and a line l which generates it as described.*

Proof. Let us introduce a polar coordinate system with a pole at O and axis on the TO-ray. The angular coordinates are measured in the counterclockwise direction; see Figure 4. Let azimuth of X be $\phi\in[0,\frac{\pi}{2})\cup(\frac{3\pi}{2},2\pi]$. If $\phi\in[0,\frac{\pi}{2})$, then $|OX|=r_\omega\sec\phi$. By construction, $\triangle OX'X$ is isosceles with $|OX'|=|OX|$. Since $\triangle OTX=\triangle ORX=\triangle ORX'$, we have $\measuredangle TOX'=3\phi$. Thus, X' has polar coordinates $(r_\omega\sec\phi,3\phi)$, which gives the polar equation $r=r_\omega\sec\frac{\theta}{3}$. If $\phi\in(\frac{3\pi}{2},2\pi]$, we get the same equation after replacing ϕ in our analysis by $\phi'=2\pi-\phi$. Note that $t(\omega,l)$ has an asymptote which is perpendicular to the polar axis and passes over the point $(2\pi,3r_\omega)$.

Conversely, suppose $t(\omega,l)$ is a curve with polar equation $r=a\sec\frac{\theta}{3}$. Then ω is chosen as the circle with centre at the pole and radius a, and l is tangent to ω at point $T(a,0)$. We can repeat the above argument to show that this configuration generates $t(\omega,l)$. ■

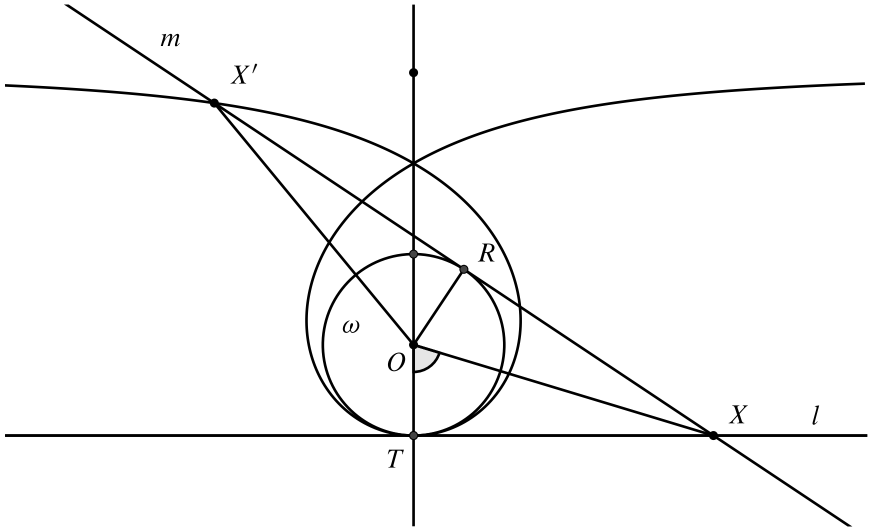


Figure 4. A characterization of the Maclaurin trisectrix.

Remarks. Another derivation of this curve and some motivations for finding alternative derivations of classical curves are given in [21].

Let us now prove our first main result.

Theorem 4. *If P is a convex k-gon containing $n \in \mathbb{N}$ non-overlapping unit disks, then*

$$\|P\| \geq \sqrt{12} \cdot (n-1) + (2-\sqrt{3}) \cdot \lceil \sqrt{12n-3} - 3 \rceil + k \cdot \tan \frac{\pi}{k}.$$

Equality holds if and only if the disks are located in a Wegner packing, P is equiangular, each side of P is tangent to at least one of the disks and each cap of P with respect to the convex hull of the disks is a cap with respect to a unit disk.

Proof. By Theorem 3, we may assume that each side of P is tangent to the convex hull of the disks, which we denote as H. This is equivalent to assume that each side is tangent to at least one of the disks. Since $\|P\| = \|H\| + \|P \setminus H\|$, by Theorem 2, it suffices to prove that

$$\|P \setminus H\| \geq k \cdot \tan \frac{\pi}{k} - \pi \tag{1}$$

where the right-hand side is the sum of cap areas of a unit k-gon with respect to the circumscribed unit disk, while the left-hand side is $\|\mathrm{Cap}_H(P)\|$.

Let ω be one of the disks and P' the shrink of P for ω; see Figure 5 (a). Since P' is a circumscribing convex k-gon of ω, by Theorem 1,

$$\|\mathrm{Cap}_\omega(P')\| \geq k \cdot \tan \frac{\pi}{k} - \pi \tag{2}$$

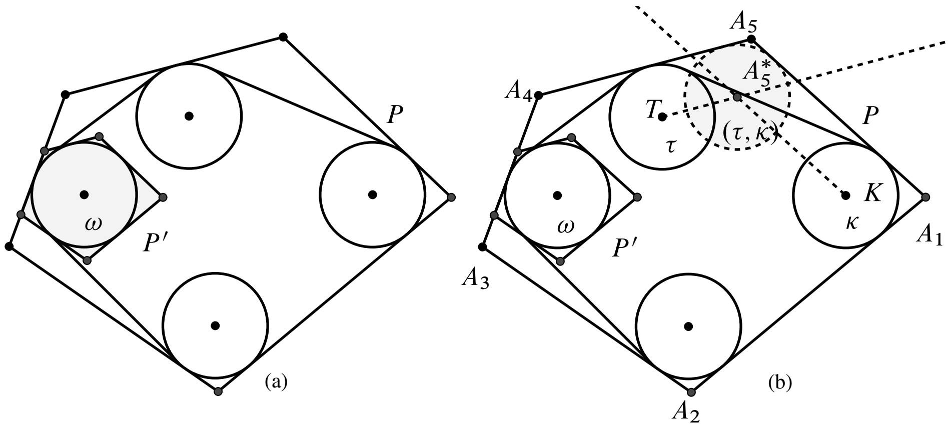


Figure 5. Shrinking P to P' in (a); adding an auxiliary disk in (b).

and equality holds if and only if P' is regular.

We claim that each cap of P' with respect to ω is smaller (in area) than the corresponding cap of P with respect to H. To see this, let $A_1, \ldots, A_k$ be the vertices of P, and let $\mathbb{S}$ be the set of n disks that it contains. Choose one of the vertices, A_i. If $A_{i-1}A_i$ and A_iA_{i+1} are tangent to the same unit disk in $\mathbb{S}$, then the cap of P with respect to H with a vertex at A_i is a cap with respect to a unit disk. Since shrinking preserves the internal angle at A_i, the corresponding cap of P' is a translation of this cap; see Figure 5 (b). Thus, they are congruent.

Otherwise, $A_{i-1}A_i$ and A_iA_{i+1} are tangent to two different unit disks in $\mathbb{S}$. Let they be τ with centre T and κ with centre K, respectively. Draw a line passing through T which is parallel to $A_{i-1}A_i$ and another line passing through K which is parallel to A_iA_{i+1} as in Figure 5 (b). Let A_i^* be their intersection. Note that this point is well defined; on the bisector of $\angle A_i$ and because of convexity, it is inside P and closer to A_i than both T and K. Let (τ, κ) be the unit disk centred at A_i^*. By construction, (τ, κ) is tangent to $A_{i-1}A_i$ and A_iA_{i+1}. If H' is the convex hull of $\mathbb{S} \cup \{(\tau, \kappa)\}$, then the cap of P with respect to H' with a vertex at A_i is the same as the corresponding cap of P' with respect to ω by the above argument. But the former cap is smaller than the cap of P with respect to H with a vertex at A_i by the amount

$$\|H'\| - \|H\| = \|\triangle A_i^*TK\| + |TA_i^*| + |A_i^*K| - |TK| > 0$$

(recall the triangle inequality). This proves our claim, which implies that

$$\|\mathrm{Cap}_H(P)\| \geq \|\mathrm{Cap}_\omega(P')\| \tag{3}$$

and equality holds if and only if each cap of P with respect to H is a cap with respect to a unit disk. Inequalities (2) and (3) imply (1).

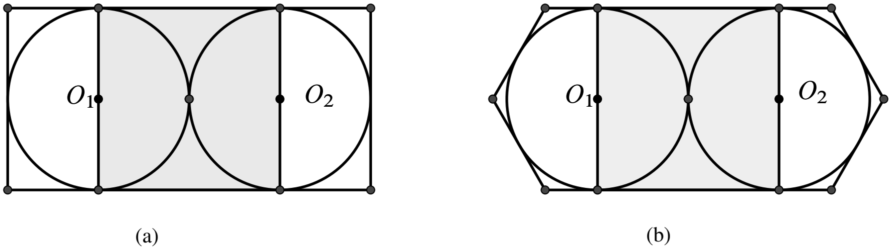


Figure 6. Construction of optimal polygons for $n = 2$ and $k = 4$ in (a), $k = 6$ in (b).

From the proof above, it should be clear that equality holds if and only if

- each side of P is tangent to at least one of the disks,
- $\|H\| = \sqrt{12} \cdot (n-1) + (2-\sqrt{3}) \cdot \lceil \sqrt{12n-3} - 3 \rceil + \pi$, which happens if and only if the disks constitute a Wegner packing by Theorem 2,
- each cap of P with respect to the convex hull of the disks is a cap with respect to a unit disk,
- P' is regular, which happens when P is equiangular (recall that P' and P have the same internal angles). ■

Remarks. Above, we used Theorem 2 to prove Theorem 4. One should notice that the reverse implication is also possible and very much the same.

The following result shows that the above inequality is tight in many cases.

Theorem 5. *If*

(a) $n = 2$ *and* $k = 2k'$ *with* $k' \geq 2$, *or*

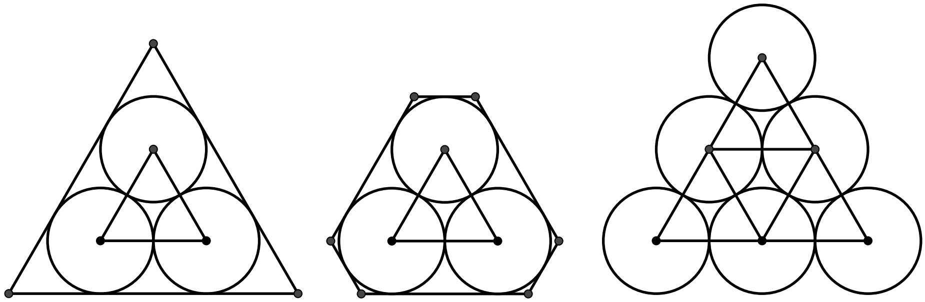
(b) $n \in \{3, 6\}$ *and* $k = 3k'$, *or*

(c) $n \in \mathbb{N} \setminus \{2\}$ *is not exceptional and* $k = 6k'$,

then the inequality in Theorem 4 is tight and the solution of Problem 1 can be constructed. In particular, when $n = 3m(m-1) + 1$ *and* $k = 6$, *the solution is the regular hexagon with sides of length* $2(m-1) + \frac{2}{\sqrt{3}}$.

Proof. Let us prove (a). Two unit disks are Wegner packed if and only if they are tangent. Let O_1, O_2 be the centres of the disks, and let us construct the solution of Problem 1 as follows. First, draw the tangent polygon of the disks; recall that in this case it is an infinite strip. Cut out a rectangle whose two opposite sides are contained in the joint tangents of the disks, and the other two sides pass through O_1 and O_2. For each of these latter two sides, take half of a unit $2k'$-gon, and paste (i.e. glue) it over its side of length 2 with the rectangle. The resulting $2k'$-gon contains the two disks and satisfies all the equality requirements in Theorem 4; see Figure 6.

Let us prove (b). Let $n = 3$, and it is easy to see that centres of three Wegner packed disks constitute vertices of an equilateral triangle with sides of length 2 as in Figure 7 (a). Then their tangent polygon T_1 satisfies all the equality conditions in Theorem 4. Thus, T_1



(a) (b) (c)

Figure 7. Wegner packing and optimal polygons when $n = 3, 6$.

is the solution when $k' = 1$. For $k' = 2$, we can cut from T_1 three equilateral triangles; each has a vertex common with T_1 and a side tangent to the convex hull of the circles. The remaining hexagon which we denote by T_2 satisfies the necessary conditions; see Figure 7 (b). In general, $T_{k'}$ is constructed as follows. Remove all caps of T_1 with respect to the convex hull of the disks, and replace each by the union of caps of one-third of the unit $3k'$-gon with respect to the circumscribed unit disk. It is easy to check that this construction is well defined and satisfies the necessary conditions.

If $n = 6$, a similar argument proves the statement after noting that centres of six Wegner packed disks constitute vertices of an equilateral triangle with sides of length 4 as in Figure 7 (c).

Let us prove (c). Since n is not exceptional, there is a Wegner packing of n disks. Because of the perimeter condition, a linear packing of n disks is never a Wegner packing for $n \neq 2$. Thus, the convex hull of the centres of Wegner packed $n \neq 2$ disks which we denote as T_c has at least three sides. Let T be the tangential polygon of the packed disks. Since T_c has at least three and at most six sides and is triangulated into equilateral triangles, and T_c and T are parallel, T has three to six sides, and its internal angles are either $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ (recall Proposition 1 (b)). This gives us the following possibilities:

- T is an equilateral triangle, or
- T is a quadrilateral with two internal angles of $\frac{2\pi}{3}$ and two of $\frac{\pi}{3}$, or
- T is a pentagon with four internal angles of $\frac{2\pi}{3}$ and one of $\frac{\pi}{3}$, or
- T is a hexagon with internal angles of $\frac{2\pi}{3}$.

In each case, take a vertex with angle $\frac{\pi}{3}$, and cut out an equilateral triangle from T which shares this vertex and the side of it which does not contain this vertex is tangent to the disk closest to the vertex. Let the resulting polygon be T_1. Then, by construction, T_1 is a convex hexagon containing all the disks; each of its internal angles is $\frac{2\pi}{3}$, and each side is tangent to at least one of the disks, and each cap is a cap with respect to a unit disk. Thus, T_1 satisfies all equality conditions in Theorem 4 and is a solution to Problem 1 when n is not exceptional and $k = 6$.

In general, $T_{k'}$ is constructed as follows. Remove all six caps of T_1 with respect to the convex hull of the disks, and replace each by the union of caps of one sixth of the unit $6k'$-gon with respect to the unit disk which it circumscribes. It is easy to check that this construction is well defined and satisfies the equality conditions.

If $n = 3m(m-1)+1$, i.e. the centred hexagonal number, it is easy to verify that the Wegner packing of n disks is so that the convex hull of their centres is a regular hexagon with sides of length $2m$. Then their tangent polygon is the regular hexagon of sides of length $2(m-1)+\frac{2}{\sqrt{3}}$ and satisfies all the equality conditions in Theorem 4. Thus, the tangent hexagon is the solution. ∎

Remarks. With some effort, one can show that a construction similar to that in Theorem 5 (a) works for $n = 2$ and $k = 2k'+1$. This time, we need to paste half of a unit $2k'$-gon and half of a unit $2(k'+1)$-gon to the central rectangle. Moreover, this reasoning can be applied to finding the smallest convex k-gon containing n linearly packed disks. In [17], we used insights from this result to mathematically investigate features of a cultural artefact from the National Museum of Mongolia.

So far, our solutions for Problem 1 relied on the cases where the disks are Wegner packed, i.e. efficiently packed. The following result shows that this is not always the case.

Theorem 6. *Let $MNKL$ be the smallest-area convex quadrilateral containing three efficiently packed unit disks, and let P^* be the 2×6 rectangle in which the disks are packed linearly. Then $\|MNKL\| > \|P^*\|$.*

Proof. Let ω_i, $i = 1, 2, 3$, be the disks and O_i their centres. We know that $|O_1O_2| = |O_2O_3| = |O_3O_1| = 2$. Let $\triangle ABC$ be the tangent polygon of the disks such that A, B are on the joint tangent of ω_1, ω_2, B, C are on the joint tangent of ω_2, ω_3 and C, A are on the joint tangent of ω_3, ω_1. Let $\omega_1 \cap AB = T_1$, $\omega_2 \cap AB = T_2$, $\omega_2 \cap BC = T_3$, $\omega_3 \cap BC = T_4$, $\omega_3 \cap AC = T_5$ and $\omega_1 \cap AC = T_6$. Further, let $\triangle AA'A''$ be such that $A' \in AB$, $A'' \in AC$ and ω_1 is an excircle tangent to $A'A''$ on its midpoint. Let $\triangle BB'B''$ and $\triangle CC'C''$ be defined analogously; see Figure 8.

Let us draw the section of the Maclaurin trisectrix for ω_1 and the AB-line ranging between A'' and the reflection A with respect to T_6. We call this curve the trisectrix for (ω_1, AA'). Draw similarly trisectrices for (ω_1, AA''), (ω_2, BB'), (ω_2, BB''), (ω_3, CC') and (ω_3, CC''). Let I_1, I_2, I_3 denote pairwise intersections of these six curves. Let $R_1 \in (AA')$ be such that R_1I_3 is tangent to ω_1 by its midpoint. We define other points R_i, $2 \le i \le 6$, analogously. Let r_{ij} with $i, j \in \{1, 2, 3\}$ and $i < j$ denote the radical axis of ω_i and ω_j, and let O be their common intersection. Because of symmetry in our configuration, $I_1 \in r_{12}$, $I_2 \in r_{23}$, $I_3 \in r_{13}$. Let $r_{12} \cap AB = P_1$ and $r_{23} \cap BC = P_2$.

Claim. $MNKL \in \{A'BCA'', B'CAB'', C'ABC'', I_1R_3CR_6, I_2R_5AR_2, I_3R_1BR_4\}$.

We assume that $MNKL$ is clockwise oriented. Let H be the convex hull of $\omega_1, \omega_2, \omega_3$. Then

$$\|H\| = \pi + 6 + \sqrt{3}. \tag{4}$$

By Theorem 3, we know that each side of $MNKL$ is tangent to H on its midpoint. This implies that each side is tangent to at least one of the disks. Since there are four sides and

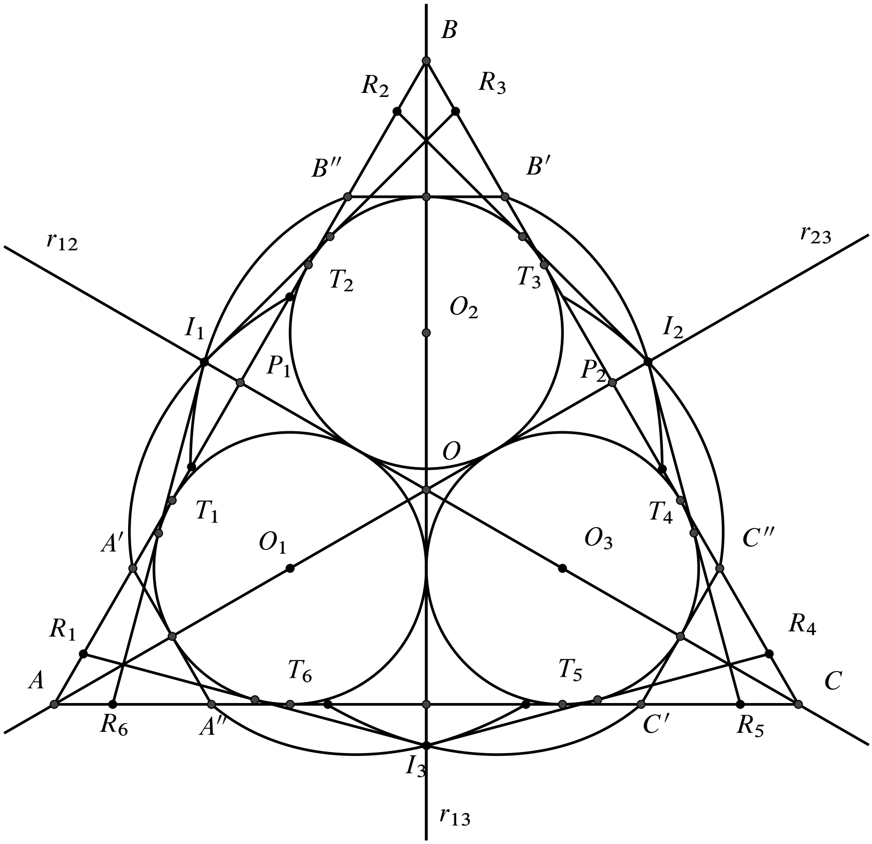


Figure 8. Tangent polygon of the three disks.

three disks, one disk must be tangent to at least two consecutive sides. Let these be MN and ML. This implies $M \in \triangle ABC$. Let us first assume that M is located in the A-vertex cap of $\triangle ABC$, i.e. MN, ML are tangent to ω_1. We shall analyse the following cases.

(a) Assume $M = A'$. Then (by Theorem 3) $L = A''$, N is on the $A'B$-ray, while K is on the $A''C$-ray. Since NK must be tangent to H, this implies $N = B$ and $K = C$. So we end up with $A'BCA''$. The case of $M = A''$ is treated analogously.

(b) Assume M is located in the A'-vertex cap of $A'BCA''$ with respect to ω_1. Then L is located either in the same cap or in $\triangle AA'A''$, while N is either in the same cap or in the region bounded by the trisectrix for (ω_1, AA'') and $A'T_2$, or on the T_2B-ray. The location of L implies that K must be either in the A-vertex cap or in the region bounded by the trisectrix for (ω_1, AA') and AT_5. In all cases, NK necessarily cuts at least one of the disks. Thus, M cannot be in the A'-vertex cap of $A'BCA''$. Similarly, it cannot be in the A''-vertex cap of $A'BCA''$.

(c) Assume $M \in \text{int}(\triangle AA'A'')$. Then N is located in one of the A'-vertex caps or the interior of the region bounded by the trisectrix for (ω_1, AA'') and $A'T_2$. The former is not possible by (b). Then L is located in one of the A''-vertex caps or the interior of the region bounded by the trisectrix for (ω_1, AA') and $A''T_5$. Again, the former is not possible. The position of N implies K is either in the B''-vertex cap or above the $B''B'$-line. The former case is not possible by (b). Similarly, the position of L implies K is located below the $C'C''$-line. Let Q be the intersection of the $B'B''$-line and the $C'C''$-line. Our last two conclusions imply that the K-vertex cap of $MNKL$ contains $\triangle B'QC''$, whose area is

$$\|\triangle B'QC''\| = \frac{4+\sqrt{12}}{\sqrt{3}}. \tag{5}$$

Notice that $\|MNKL\| > \|H\| + \|\triangle B'QC''\|$. On the other hand,

$$\|A'BCA''\| = \frac{11+6\sqrt{3}}{\sqrt{3}}. \tag{6}$$

By using (4), (5), (6), we know

$$\|MNKL\| - \|A'BCA''\| > \|H\| + \|\triangle B'QC''\| - \|A'BCA''\| > 0.$$

Thus, $MNKL$ is not the smallest.

(d) Assume $M \in (R_1A')$. Then L is on the section strictly between A'' and I_3 of the trisectrix for (ω_1, AA'), while N is on the $A'B$-ray. The former conclusion implies K is located above the BC-line and below the I_3R_4-line. Then NK can only be tangent to ω_2. This in turn implies $N \in (T_2B)$. Then the distance from N to a point on the T_2T_3-arc of ω_2 is much smaller than the distance from K to the same point. In particular, the former distance is at most $|BT_3| = \sqrt{3}$, while the latter distance is greater than $|T_3T_4| = 4$ (to see this, just project K to BC). Thus, such $MNKL$ can not be the smallest. Similarly, $M \in (R_6A'')$ gives a non-optimal solution.

(e) Assume $M \in (AR_1)$. Then L is in the interior of the region bounded by the trisectrix for (ω_3, CC''). This implies $K \in \mathrm{int}(\triangle CC'C'')$. We can repeat the argument in (c) to reach a contradiction. Similarly, $M \in (AR_6)$; then we reach to a contradiction.

(f) Assume $M = R_1$. Then $L = I_3$, which implies $K = R_4$. These imply $MNKL = R_1BR_4I_3$. Similarly, $M = R_6$ implies $MNKL = R_6I_1R_3C$.

(g) Assume $M = A$. Then N is on the T_2B-ray and L is on the T_5C-ray. Since both NK and KL are tangent to H, we must have either $L \in T_5C$ or $N \in T_2B$. First, assume $L \in T_5C$. If $L \in (T_5C)$, then by the arguments above, we know that either $L = C'$ or $L = R_5$. In the first case, we end up with $ABC''C'$ and in the second case with $AR_2I_2R_5$. If $L = C$, then $MNKL = AB''B'C$. Similarly, $N \in (T_2B)$ implies

$$MNKL \in \{B'CAB'', I_2R_5AR_2, ABC''C'\}.$$

Thus, we conclude that, when M is located in the A-vertex cap of $\triangle ABC$,

$$MNKL \in \{A'BCA'', B'CAB'', C'ABC'', I_1R_3CR_6, I_2R_5AR_2, I_3R_1BR_4\}.$$

A similar argument shows that $MNKL$ is one of these six polygons when M is located in the C-vertex cap or B-vertex cap of $\triangle ABC$. This proves our claim.

Symmetries in our configuration imply $\|A'BCA''\| = \|B'CAB''\| = \|C'ABC''\|$ and $\|I_1R_3CR_6\| = \|I_2R_5AR_2\| = \|I_3R_1BR_4\|$. Then

$$\|A'BCA''\| = \frac{11+6\sqrt{3}}{\sqrt{3}} > 12 = \|P^*\|.$$

Notice that $\|I_3R_1BR_4\| = \|R_1P_1OI_3\| + \|P_1BP_2O\| + \|P_2CI_3O\|$. Since each of these three quadrilaterals contains a unit disk and is non-regular, each has area greater than 4 by Theorem 1. Then $\|I_3R_1BR_4\| > 12 = \|P^*\|$. ■

Remarks. By now, we know that $n = 3, k = 4$ is the first case where the disks are packed non-efficiently inside the smallest containing polygon, when we order (n, k) lexicographically.

4 Final discussions

Let us discuss some open problems. Because of Theorem 6, packing of n disks in the smallest convex k-gon is not always the most efficient. However, by the Dowker inequality [8], at any fixed location of the disks, the area of the smallest containing k-gon tends rather fast to the area of its convex hull as $k \to \infty$. This is a supportive fact for the efficient packing to realize inside the smallest containing k-gon when k is large. It is believed that efficient packing of $n \in \mathbb{N}$ unit disks is a Groemer packing [5]. We can then ask whether the packing of n disks in the smallest convex k-gon is always a Groemer packing?

The following assertion seems very plausible.

Conjecture 1. *If $n = \frac{m(m+1)}{2}$, i.e. the m-th triangular number, then the smallest triangle containing n unit disks is the equilateral triangle of side $2(m-1) + 2\sqrt{3}$.*

In Theorem 1 and Theorem 5 (b), we proved it for $m = 1, 2, 3$, but our approach stopped at $m = 4$, where the Wegner packing of 10 disks is not triangular; see Figure 3. One can show that the smallest triangle containing two unit disks is the isosceles right triangle with hypotenuse of length $6 + 4\sqrt{2}$, i.e. it is not equilateral. However, it is plausible that the smallest triangle containing $\frac{m(m+1)}{2} - 1$ unit disks with $m > 2$ is equilateral. These lead to a new version of the long standing Erdős–Oler conjecture [18].

Conjecture 2. *If $n > 3$ is a triangular number, then the smallest triangle containing n disks is the same as that containing $n - 1$ disks.*

In Theorem 5, we showed that if n is a centred hexagonal number, then the smallest containing hexagon is regular. Then we can propose the following analogy of Conjecture 2.

Conjecture 3. *If $n \in \mathbb{N}$ is a centred hexagonal number, then the smallest hexagon containing n disks is the same as that containing $n - 1$ disks.*

The following assertion would simplify some of our proofs. Let P be the smallest convex k-gon containing a convex disk C. Let P^1 be the convex polygon obtained after cutting the largest (area) triangles from each cap of P so that P^1 still contains C. Is it true that minimality of P implies minimality of P^1? The answer is positive if C is a unit disk, negative if C is the convex hull of two tangent unit disks and $k = 3$, but positive in the latter case when $k = 4$ (see Theorem 5 (a)). What if C is an ellipse? Moreover, does minimality of both P and P^1 imply minimality of the rest of members of the sequence obtained by repeated application of the "greedy cut" procedure?

Finally, we are inclined to think that area and perimeter are close relatives, which raises the question of solving Problem 1 for perimeter. While we can obtain a lower bound for the perimeter from Theorem 4 by the isoperimetric inequality, such a bound is too loose. The technique that we used in Section 3 is not applicable to the perimeter, as the perimeter of a container is not separable into that of a convex hull of the disks and the remaining regions. However, we hope to solve this problem in the future.

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