Elem. Math. (Online first)
DOI 10.4171/EM/449

Elemente der Mathematik

Short note On the universality of Somos' constant

Jörg Neunhäuserer

Abstract. We show that Somos' constant is universal in a sense that is similar to the universality of the Khinchin constant. In addition, we introduce generalized Somos' constants, which are universal in a similar sense.

1 Introduction and main result

Let us first recall the definition of the Khinchin constant

$$K = \prod_{i=1}^{\infty}\Big(1 + \frac{1}{i(i+2)}\Big)^{\log_2 i} = 2.6854520010\ldots.$$

By the famous theorem of Khinchin [3], this constant is universal in the following sense: For almost all real numbers x, the geometric mean of the entries of the continued fractions of x converges to K. We consider here Somos' constant

$$\sigma = \prod_{i=1}^{\infty} \sqrt[2^i]{i} = 1.6616879496\ldots,$$

which first appeared in [7] in the context of the quadratic recurrence $g_n = n g_{n-1}^2$, see also [1, page 446]. In the recent past, this constant raised some attention, see for instance [2, 4, 6]. We will show that the Somos constant is universal in a sense that is similar to the universality of the Khinchin constant. In [5], we represent real numbers $x \in (0,1]$ in the form

$$x = \langle n_1, n_2, n_3, \ldots\rangle := \sum_{k=1}^{\infty} 2^{-(n_1+n_2+\cdots+n_k)}$$

with $n_k \in \mathbb{N}$ and show that the representation is unique. Replacing the continued fraction representation by this representation, we obtain the universality of Somos' constant.

Theorem 1.1. *For almost all* $x = \langle n_1, n_2, n_3, \ldots\rangle \in (0,1]$, *we have*

$$\lim_{i\to\infty} \sqrt[i]{n_1 n_2 \ldots n_i} = \sigma.$$

We will prove this theorem in the next section. In the last section, we will introduce generalized Somos constants, which are universal with respect to a modification of the representation used here.

2 Proof of the main result

Consider the map $T\colon (0,1] \to (0,1]$, given by $T(x) = 2^i x - 1$ for $x \in (\frac{1}{2^i}, \frac{1}{2^{i-1}}]$ and $i \in \mathbb{N}$. The relation of this transformation to the expansion of real numbers, defined in the last section, is given by the following lemma.

Lemma 2.1. *Let $x = \langle n_1, n_2, n_3, \ldots \rangle \in (0,1]$. For all $k \in \mathbb{N}$, we have $T^{k-1}(x) \in (\frac{1}{2^i}, \frac{1}{2^{i-1}}]$ if and only if $n_k = i$.*

Proof. Obviously, $T(\langle n_1, n_2, n_3, \ldots \rangle) = \langle n_2, n_3, n_4, \ldots \rangle$. Since $x \in (\frac{1}{2^i}, \frac{1}{2^{i-1}}]$ if and only if $n_1 = i$, the result follows immediately. ■

To apply Birkhoff's ergodic theorem, we prove the following.

Proposition 2.1. *The Lebesgue measure $\mathfrak{L}$ is ergodic with respect to T.*

Proof. For an open interval $(a,b) \subseteq [0,1]$, we have

$$\begin{aligned}
\mathfrak{L}(T^{-1}((a,b))) &= \mathfrak{L}\Big(\bigcup_{i=1}^{\infty}\Big(\frac{a}{2^i} + \frac{1}{2^i}, \frac{b}{2^i} + \frac{1}{2^i}\Big)\Big) \\
&= \sum_{i=1}^{\infty} 2^{-k}\mathfrak{L}\Big(\Big(\frac{a}{2^i} + \frac{1}{2^i}, \frac{b}{2^i} + \frac{1}{2^i}\Big)\Big) \\
&= \sum_{i=1}^{\infty} 2^{-i}(b-a) = b - a = \mathfrak{L}((a,b)).
\end{aligned}$$

Hence $\mathfrak{L}(T^{-1}(B)) = \mathfrak{L}(B)$ for all Borel sets $B \subseteq (0,1]$, which means that $\mathfrak{L}$ is invariant under T. Let B be a Borel set with $\mathfrak{L}(B) < 1$, which is invariant under T; that is $T(B) = B$. Note that, for all $k \in \mathbb{N}$, the intervals of the form

$$I_{m_1,\ldots,m_k} = \{\langle n_1, n_2, n_3, \ldots \rangle \mid n_i = m_i \text{ for } i = 1, \ldots, k\}$$

build a partition of $(0,1]$, where the length of the partition elements is bounded by $\frac{1}{2^k}$. By Lebesgue's density theorem, for every $\epsilon > 0$, there is an interval $I = I_{m_1,\ldots,m_k}$ such that $\mathfrak{L}(I \backslash B) \geq (1-\epsilon)\mathfrak{L}(I)$. Since $T^k(I) = (0,1]$, we have

$$\mathfrak{L}((0,1] \backslash B) \geq \mathfrak{L}(T^k(I \backslash B)) \geq (1-\epsilon)\mathfrak{L}(T^k(I)) = 1 - \epsilon.$$

Hence $\mathfrak{L}(B) = 0$. This proves that μ is ergodic. ■

Now we are prepared to prove Theorem 1.1. Let

$$f(x) = \sum_{i=1}^{\infty} \log(i)\chi_{(1/2^i, 1/2^{i-1}]}(x),$$

where χ is the characteristic function. By Lemma 2.1, we have $f(T^{k-1}(x)) = \log(n_k)$ for $x = \langle n_1, n_2, n_3, \ldots \rangle$. Applying Birkhoff's ergodic theorem to T with the L^1-function f, we obtain

$$\lim_{i\to\infty}\frac{1}{i}\sum_{k=1}^{i}\log(n_k)=\lim_{i\to\infty}\frac{1}{i}\sum_{k=1}^{i}f(T^{k-1}(x))=\int_0^1 f(x)\,dx=\sum_{i=1}^{\infty}\log(i)2^{-i}$$

for almost all $x=\langle n_1,n_2,n_3,\dots\rangle\in(0,1]$. Taking the exponential gives the result.

3 A generalization

Let $b\geq 2$ be an integer. It is easy to show that a real number $x\in(0,1]$ has a unique representation in the form

$$x=\langle n_1,n_2,n_3,\dots\rangle_b:=(b-1)\sum_{k=1}^{\infty}b^{-(n_1+n_2+\cdots+n_k)}$$

with $n_k\in\mathbb{N}$; the argument can be found in [5]. Now consider the map $T_b\colon(0,1]\to(0,1]$, given by $T_b(x)=b^i x-(b-1)$ for $x\in((b-1)b^{-i},(b-1)b^{1-i}]$ and $i\in\mathbb{N}$. Using the argument in the last section with respect to T_b instead of T, we obtain the following.

Theorem 3.1. *For almost all* $x=\langle n_1,n_2,n_3,\dots\rangle_b\in(0,1]$, *we have*

$$\lim_{i\to\infty}\sqrt[i]{n_1n_2\dots n_i}=\prod_{i=1}^{\infty}\sqrt[b^i]{i^{b-1}}=:\sigma_b.$$

We call σ_b for $b>2$ a generalized Somos constant. These constants are universal with respect to the base b representation $\langle n_1,n_2,n_3,\dots\rangle_b$. The generalization given here is slightly different from the generalization of Somos' constant studied in [8], which is not related to universality.[1]

We end the paper with a nice expression of generalized Somos constants σ_b using values of the generalized Euler-constant function

$$\gamma(z)=\sum_{i=1}^{\infty}z^{i-1}\Big(\frac{1}{i}-\log\Big(\frac{i+1}{i}\Big)\Big),$$

where $|z|\leq 1$.

Proposition 3.1. *For all integers* $b\geq 2$, *we have*

$$\sigma_b=\frac{b}{b-1}e^{-\gamma(1/b)/b}.$$

[1] They consider $\sqrt[b-1]{\sigma_b}$.

Proof. We have

$$\gamma\Big(\frac{1}{b}\Big) = \sum_{i=1}^{\infty}\Big(\frac{b^{-i+1}}{i} - b^{-i+1}\log(i+1) + b^{-i+1}\log(i)\Big)$$
$$= b\Big(\sum_{i=1}^{\infty}\frac{b^{-i}}{i} - \sum_{i=1}^{\infty} b^{-i}\log(i+1) + \sum_{i=1}^{\infty} b^{-i}\log(i)\Big)$$
$$= b\Big(\log\Big(\frac{b}{b-1}\Big) - \frac{b\sigma_b}{b-1} + \frac{\sigma_b}{b-1}\Big) = b\log\Big(\frac{b}{(b-1)\sigma_b}\Big)$$

using

$$\sum_{i=1}^{\infty}\frac{b^{-i}}{i} = \log\Big(\frac{b}{b-1}\Big) \quad \text{and} \quad \log(\sigma_b) = \sum_{i=1}^{\infty} b^{-i}(b-1)\log(i).$$

Hence

$$e^{\gamma(1/b)} = \Big(\frac{b}{(b-1)\sigma_b}\Big)^b \quad \text{and} \quad e^{\gamma(1/b)/b} = \frac{b}{(b-1)\sigma_b} \quad \text{given} \quad \sigma_b = \frac{be^{-\gamma(1/b)/b}}{b-1}. \quad \blacksquare$$

Estimates of $\gamma(\frac{1}{b})$ can be found in [4].

References

[1] S. R. Finch, *Mathematical Constants*, Cambridge University Press, Cambridge, 2003.

[2] J. Guillera and J. Sondow, Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent, *Ramanujan J.* **16** (2008), no. 3, 247–270.

[3] A. Khinchin, Metrische Kettenbruchprobleme, *Compos. Math.* **1** (1935), 361–382.

[4] C. Mortici, Estimating the Somos' quadratic recurrence constant, *J. Number Theory 130, no.* **12** (2010), 2650–2657.

[5] J. Neunhäuserer, On the Hausdorff dimension of fractals given by certain expansions of real numbers, *Archiv der Mathematik* **97** (2011), no. 5, 459–466.

[6] N. J. A. Sloane, Sequences A052129, A112302, A114124, and A116603 in the On-Line Encyclopedia of Integer Sequences.

[7] M. Somos, Several constants related to quadratic recurrences, unpublished note (1999).

[8] J. Sondow and P. Hadjicostas, The generalized Euler-constant function $\gamma(z)$ and a generalization of Somos's quadratic recurrence constant, *J. Math. Anal. Appl.* **332** (2007), no. 1, 292–314.

Jörg Neunhäuserer
Reitststallweg 9
38640 Goslar, Germany
joerg.neunhaeuserer@web.de