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Short note

A note on the Diophantine equation $(x+1)^3+(x+2)^3+\cdots+(2x)^3=y^n$

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Abstract. In this short note, we show that the equation in the title has no integer solutions $x, y \geq 1$ and $n > 1$.

1 Introduction

Let k, l, n be fixed positive integers. The equation

$$(x+1)^k+(x+2)^k+\cdots+(lx)^k=y^n \tag{1}$$

has been studied by many authors. Bai and Zhang [1] solved equation (1) in the case $l=k=2$. Bérczes, Pink, Savas, and Soydan [3] showed that (1) has no solutions if $l=2$, $2 \leq x \leq 13$, $y \geq 2$, and $n \geq 3$. Soydan [4] showed that (1) only has a finite number of integer solutions $x, y \geq 1$ if $k \neq 1, 3$ and $n \geq 2$. Bartoli and Soydan [2] showed that every positive integer solutions x, y of (1) must satisfy $\max\{x, y, n\} < C$, where C is a computable constant depending only on k, l. In this short note, we completely solve equation (1) when $l=2$, $k=3$, and $n \geq 2$. In fact, we will give an elementary proof of the following theorem.

Theorem 1. *Let $n \geq 2$ be a positive integer. Then the equation*

$$(x+1)^3+(x+2)^3+\cdots+(2x)^3=y^n \tag{2}$$

has no integer solutions $x, y \geq 1$.

2 A proof of Theorem 1

Assume there exist integers $x, y \geq 1$ satisfying (2). Using the formula

$$1^3+2^3+\cdots+m^3=\frac{m^2(m+1)^2}{4} \quad \text{for all } m \in Z^+,$$

equation (2) is equivalent to

$$y^n=\frac{(2x)^2(2x+1)^2}{4}-\frac{x^2(x+1)^2}{4}=\frac{x^2(3x+1)(5x+3)}{4}. \tag{3}$$

Case 1: *n* is even. Then, from (3), we have $(3x+1)(5x+3)$ is a perfect square. Since $\gcd(3x+1, 5x+3) \in \{1,2,4\}$, both $5x+3$ and $3x+1$ are perfect squares or two times perfect squares. The first case is impossible modulo 5, and the second case is impossible modulo 3.

Case 2: *n* is odd.

Case 2.1: x is even. Let $x = 2a$, where $a \in \mathbb{Z}^+$. Then (3) becomes

$$a^2(6a+1)(10a+3) = y^n. \tag{4}$$

If $3 \nmid a$, then $\gcd(a^2, (6a+1)(10a+3)) = 1$. Therefore, from (4), we have $a^2 = A^n$, where $A \in \mathbb{Z}^+$, which is impossible since n is odd. If $3 \mid a$, let $a = 3b$, where $b \in \mathbb{Z}^+$. Equation (4) becomes

$$3^3 b^2(18b+1)(10b+1) = y^n. \tag{5}$$

Since $\gcd(b^2, (18b+1)(10b+1)) = 1$, from (5), we have $b^2 = A^n$ or $b^2 = 3^{n-3}A^n$, where $A \in \mathbb{Z}^+$, which is also impossible since n is odd.

Case 2.2: x is odd. Let $x = 2a+1$, where $a \in \mathbb{Z}$, $a \geq 0$. Then (3) becomes

$$(2a+1)^2(3a+2)(5a+4) = y^n. \tag{6}$$

If $3 \nmid 2a+1$, then $\gcd(2a+1, 5a+4) = \gcd(2a+1, 3a+2) = 1$. Therefore, from (6), we have $(2a+1)^2 = C^n$, where $C \in \mathbb{Z}^+$, which is impossible since n is odd. If $3 \mid 2a+1$, let $a = 3b+1$, where $b \in Z$, $b \geq 0$. Equation (6) becomes

$$3^3(2b+1)^2(9b+5)(5b+3) = y^n. \tag{7}$$

Since $\gcd(2b+1, 9b+5) = \gcd(2b+1, 5b+3) = 1$, from (7), we have

$$(2b+1)^2 = C^n \quad \text{or} \quad (2b+1)^2 = 3^{n-3}C^n,$$

where $C \in Z^+$, which is impossible since n is odd. Theorem 1 is proved.

References

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