
Apparent paradoxical partitions in countable sets

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1 Introduction

Let (\mathbf{P}) be a property defined on the set of the natural numbers, and let A_n be the set of positive integers less than or equal to n , where n is a positive integer. If an integer is selected at random from A_n , then the probability σ_n that it satisfies (\mathbf{P}) is the number of elements in A_n that satisfy (\mathbf{P}) divided by n . If the limit of σ_n when n tends to infinity exists, this limit is called the natural density or asymptotic density of the integers satisfying (\mathbf{P}) . Intuitively, this density measures how frequently an integer satisfies (\mathbf{P}) . One can define analogously the natural density on any infinite subset of the set of natural numbers. Let us consider the following situation.

Man betrachte eine bestimmte Eigenschaft P der natürlichen Zahlen \mathbb{N} und eine Partition von \mathbb{N} in disjunkte, unendliche Teilmengen E_i , $i \in \mathbb{N}$, so dass in jedem E_i nur endlich viele Zahlen die Eigenschaft P besitzen. Insbesondere ist die Dichte der Zahlen mit der Eigenschaft P in jeder Teilmenge E_i gleich 0. Kann es sein, dass trotzdem die Dichte der Zahlen mit der Eigenschaft P in \mathbb{N} mindestens $1/2$ beträgt? Auf den ersten Blick scheint dies paradox, obwohl es bei genauerer Betrachtung einfach zu erklären ist. Dasselbe Phänomen zeigt sich auch im Kontext von algebraischen Zahlen: Auch dort wurde eine Klasse algebraischer Zahlen mit einer Dichte von $1/\zeta(3) = 0.8319\dots$ identifiziert, obwohl eine Zerlegung scheinbar nahelegt, dass die Zahlen selten sind. Die Autoren der vorliegenden Arbeit gehen dem Phänomen auf den Grund und formulieren interessante Anschlussfragen für die weitere Forschung.

Let E be an infinite set on which a property (\mathbf{P}) is defined. Suppose that $E = \bigcup_{i \in I} E_i$ is a partition, where each E_i is infinite. Suppose also that, in each E_i , the number of elements satisfying (\mathbf{P}) is finite. Then, clearly, the density of the elements satisfying (\mathbf{P}) is 0 in every E_i . Is it possible that the density of the subset of E containing all the elements satisfying (\mathbf{P}) will be at least equal to $1/2$?

We were first confronted with this situation while reading the paper of Arno et al. [1]. In fact, it is in the paper [1] where it is shown that the density of certain algebraic numbers in $\overline{\mathbb{Q}}$, which we will call Arno et al. numbers in Section 5, is equal to $1/\zeta(3)$. We have partitioned $\overline{\mathbb{Q}}$ in a way that suggests these Arno et al. numbers are rare. This phenomenon struck us as contradictory, which lead us to consider the situation in greater detail. We will show in the sequel, through two examples, that the answer to the above question may be positive. At first glance, this problem resembles to the so called Simpson paradox in probability and statistics. In this paper, when we say the density, we mean the natural density.

2 First example

Let $E = \mathbb{N} \setminus \{0, 1\}$. In this set, the relation (\mathbf{P}) will be that x is even. For any $i \geq 1$, let

$$E_i = \{2i\} \cup \{2^i k + 1, k \geq 1 \text{ odd}\}.$$

The E_i constitute a partition of E ; each class contains a unique element satisfying (\mathbf{P}) . On the other hand, the density in E of the elements satisfying (\mathbf{P}) is equal to $1/2$.

Before producing the second example, we need to state, in the following section, some results on the denominator and constant coefficient of an algebraic number. The reader who is familiar with these notions may skip this section.

3 Denominator of an algebraic number

Let γ be an algebraic number of degree n , and let

$$g(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

be the unique irreducible polynomial with integral coefficients such that $g(\gamma) = 0$ and $c_n > 0$. This polynomial will be called the minimal polynomial of γ over \mathbb{Z} . Let

$$I = I(\gamma) = \{k \in \mathbb{Z}, k\gamma \text{ an algebraic integer}\} \quad (1)$$

Then $I(\gamma)$ is a principal ideal of \mathbb{Z} . Since $c_n \gamma$ is an algebraic integer, then $c_n \in I(\gamma)$. Hence $I(\gamma) \neq (0)$. Let $d > 0$ be a generator of $I(\gamma)$, then d is the smallest positive integer such that $d\gamma$ is integral. Moreover, $d \mid c_n$. Call this integer d the denominator of γ . Let $\theta = d\gamma$, then θ is an algebraic integer, called the numerator of γ . The leading coefficient c_n of $g(x)$ will be called the leading coefficient of γ . Denote by $d(\gamma)$ and $c(\gamma)$ the denominator and the leading coefficient of γ respectively.

Proposition 1. *Let γ be an algebraic number of degree n , and let $\gamma_1, \dots, \gamma_n$ be the list of its conjugates over \mathbb{Q} . For any $i = 1, \dots, n$, let $s_i(\vec{\gamma}) = s_i(\gamma_1, \dots, \gamma_n)$ be the elementary symmetric function of degree i of the γ_j . Let*

$$J = J(\gamma) = \{k \in \mathbb{Z}, ks_i(\vec{\gamma}) \in \mathbb{Z} \text{ for } i = 1, \dots, n\}; \quad (2)$$

then J is a principal ideal of \mathbb{Z} generated by $c(\gamma)$. Moreover, $c(\gamma) \mid d^n$.

Proof. Let c' be a generator of $J(\gamma)$, and let $d = d(\gamma)$. Set $\gamma = \theta/d$, where θ is an algebraic integer. Since

$$d^n s_k(\gamma) = d^n \sum \gamma_{i_1} \cdots \gamma_{i_k} = d^n \sum (\theta_{i_1}/d) \cdots (\theta_{i_k}/d),$$

then $d^n \in J(\gamma)$, thus $c' \mid d^n$. Let $c = c(\gamma)$, and let $g(x)$ be the minimal polynomial of γ over \mathbb{Z} , then

$$\begin{aligned} g(x) &= c(x - \gamma_1) \cdots (x - \gamma_n) \\ &= c(x^n - s_1(\gamma)x^{n-1} + \cdots + (-1)^n s_n(\gamma)) \\ &= cx^n - cs_1(\gamma)x^{n-1} + \cdots + (-1)^n cs_n(\gamma). \end{aligned}$$

Since $g(x) \in \mathbb{Z}[x]$, then $cs_k(\vec{\gamma}) \in \mathbb{Z}$ for $k = 1, \dots, n$, hence $c \in J(\gamma)$, which implies $c' \mid c$. On the other hand, let

$$\begin{aligned} g_1(x) &= c'(x - \gamma_1) \cdots (x - \gamma_n) \\ &= c'x^n - c's_1(\gamma)x^{n-1} + \cdots + (-1)^n c's_n(\gamma) \in \mathbb{Z}[x]. \end{aligned}$$

Since $g_1(\gamma) = 0$, then $g(x)$ divides $g_1(x)$ in $\mathbb{Q}[x]$. Set $g_1(x) = g(x)q$ with $q \in \mathbb{Q}$. Using the content of polynomials, denoted by cont , we have $\text{cont}(g_1) = q \text{cont}(g) = q$, hence $q \in \mathbb{Z}$. The comparison of the leading coefficients leads to $c' = cq$, hence $c \mid c'$. Finally, we get $c = c'$. \blacksquare

Remark 2. We have shown that $d(\gamma) \mid c(\gamma)$ and $c(\gamma) \mid d(\gamma)^n$, hence $d(\gamma)$ and $c(\gamma)$ have the same prime factors.

Example 3. In the following example, we compare the denominator and the leading coefficient. In particular, we look for the condition $d(\gamma) = c(\gamma)$.

Let $\gamma = \sqrt{2}/(q2^k)$ with q positive and odd and $k \geq 0$. Here $d(\gamma) = q2^k$. Moreover, γ is a root of the equation $2^{2k}q^2\gamma^2 - 2 = 0$ which is not always irreducible over \mathbb{Z} . In the different cases which follow, we make the minimal polynomial $g(x)$ of γ over \mathbb{Z} precise.

- $k = 0$, $g(x) = q^2x^2 - 2$, hence $d(\gamma) = q$ and $c(\gamma) = q^2 > d(\gamma)$ except if $q = 1$, i.e. $\gamma = \sqrt{2}$.
- $k = 1$ and $q = 1$. Here $g(x) = 2x^2 - 1$, $d(\gamma) = c(\gamma) = 2$.
- $k = 1$ and $q \geq 2$. Here $g(x) = 2q^2x^2 - 1$, hence $d(\gamma) = 2q$, $c(\gamma) = 2q^2 > d(\gamma)$.
- $k \geq 2$ and $q \geq 1$. Here $g(x) = 2^{2k-1}q^2x^2 - 1$, hence $d(\gamma) = 2^kq$, $c(\gamma) = 2^{2k-1}q^2 > d(\gamma)$.

We thus get $c(\gamma) = d(\gamma)$ if and only if $\gamma = \sqrt{2}$ or $\gamma = \sqrt{2}/2$. This example suggests that we almost always have $c(\gamma) > d(\gamma)$.

4 Result of Arno–Robinson–Wheeler

These three authors have shown that the density in $\overline{\mathbb{Q}}$ of the algebraic numbers γ such that $c(\gamma) = d(\gamma)$ is equal to $1/\zeta(3) = 0.8319\dots$. These numbers will be called Arno et al. numbers. The height of a polynomial with integral coefficients, $P(x) = a_k x^k + \dots + a_0$, is defined by $H(P) = \max_i |a_i|$. The height of an algebraic number is the height of its minimal polynomial over \mathbb{Z} . To prove their result, Arno et al. define the sets

$$\begin{aligned}\mathcal{A}_d(H) &= \{\gamma \in \overline{\mathbb{Q}}, \deg(\gamma) = d, \text{ and } H(\gamma) \leq H\}, \\ \hat{\mathcal{A}}_d(H) &= \{\gamma \in \mathcal{A}_d(H), c(\gamma) = d(\gamma)\},\end{aligned}$$

where H is a positive integer and $H(\gamma)$ denotes the height of γ . After obtaining an asymptotic formula of $|\hat{\mathcal{A}}_d(H)|/|\mathcal{A}_d(H)|$, they deduce that

$$\lim_{d \rightarrow \infty} \lim_{H \rightarrow \infty} \left| \frac{\bigcup_{k \leq d} \hat{\mathcal{A}}_k(H)}{\bigcup_{k \leq d} \mathcal{A}_k(H)} \right| = 1/\zeta(3).$$

This result makes appear that the Arno et al. numbers are very frequent in the set of the algebraic numbers. In the sequel, we define a partition of $\overline{\mathbb{Q}}$ which suggests that the Arno et al. numbers are rare. With the characterizations of $d(\gamma)$ and $c(\gamma)$ given above, this result of Arno et al. seems surprising. Actually, $d(\gamma)$ is the smallest positive integer d such that $d\gamma$ is an algebraic integer. In contrast, $c(\gamma)$ is the smallest positive integer c such that all the products

$$c(\gamma_1 + \dots + \gamma_n), c(\gamma_1\gamma_2 + \dots + \gamma_{n-1}\gamma_n), \dots, c\gamma_1 \dots \gamma_n$$

are rational integers. For the first product, we clearly see that c may be replaced by d , but for the others? Arno's et al. result tells us that it is also true with a high probability.

5 Second example

Definition 4. Let K be a number field of degree n over \mathbb{Q} , θ an algebraic integer of K , and $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ the characteristic polynomial of θ over \mathbb{Q} . The norm of θ over \mathbb{Q} , denoted by $N_{\mathbb{Q}(\theta)/\mathbb{Q}}(\theta)$, is defined by $N_{\mathbb{Q}(\theta)/\mathbb{Q}}(\theta) = (-1)^n a_0$.

Proposition 5. Let $\gamma = \theta/d$ be an algebraic number of degree $n \geq 2$, where θ is an algebraic integer and $d = d(\gamma)$. Suppose γ is an Arno et al. number, then $d^{n-1} \mid N_{\mathbb{Q}(\theta)/\mathbb{Q}}(\theta)$.

Proof. Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

be the minimal polynomial of θ over \mathbb{Q} . Since $f(d\gamma) = 0$, we have

$$d\gamma^n + a_{n-1}\gamma^{n-1} + \dots + (a_1/d^{n-2})\gamma + (a_0/d^{n-1}) = 0.$$

The minimal polynomial of γ over \mathbb{Z} has the form

$$g(x) = dx^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0.$$

It follows that $g(x)$ divides the polynomial

$$dx^n + a_{n-1}x^{n-1} + \cdots + (a_1/d^{n-2})x + (a_0/d^{n-1}).$$

in $\mathbb{Q}[x]$. Let $q \in \mathbb{Q}$ such that

$$dx^n + a_{n-1}x^{n-1} + \cdots + (a_1/d^{n-2})x + (a_0/d^{n-1}) = qg(x).$$

Identifying the leading coefficients, we get $q = 1$, and since c_0 is an integer, $d^{n-1} \mid a_0$. ■

Make a partition of $\overline{\mathbb{Q}} \setminus \mathbb{Q}$ by putting in the same class, say $C(\theta)$, all the algebraic numbers whose numerator is equal to the algebraic integer θ . The preceding proposition shows that any class $C(\theta)$ contains a finite number (may be 0) of Arno et al. numbers. We thus have a second example of a set, namely $\overline{\mathbb{Q}} \setminus \mathbb{Q}$, and the property **(P)** states that γ is an Arno et al. number.

We have excluded from $\overline{\mathbb{Q}}$ the rational numbers because they are Arno et al. numbers and any corresponding class $C(\theta)$ is infinite.

One may think that the Arno et al. numbers are mostly frequent when their degrees are large enough. But, let $\gamma = \theta/d$ be an Arno et al. algebraic number of degree n . Let q be a positive integer divisible by some prime number p such that $p \nmid N_{\mathbb{Q}(\theta)/\mathbb{Q}}(\theta)$, then, by the preceding proposition, the number $\gamma_q = \theta/q$ has the same degree as γ and generates the same field, but it is not an Arno et al. number. Obviously, the number of γ_q 's is infinite.

6 Denominator and leading coefficient in the formal case

Proposition 6. *Let t_0, t_1, \dots, t_n, x be algebraically independent variables over \mathbb{Z} , and let*

$$G(x) = t_n x^n + \cdots + t_1 x + t_0 \in \mathbb{Z}[t_0, t_1, \dots, t_n][x].$$

Set $\vec{t} = (t_0, t_1, \dots, t_n)$. Let Γ be a root of $G(x)$ in an algebraic closure of $\mathbb{Q}(\vec{t})$. Let

$$I = \{D(\vec{t}) \in \mathbb{Z}[\vec{t}], D\Gamma \text{ is integral over } \mathbb{Z}[\vec{t}]\};$$

then I is a principal ideal of $\mathbb{Z}[\vec{t}]$ generated by t_n .

Proof. Obviously, $t_n \in I$. Let $D(\vec{t}) \in I$, then, since

$$t_n \Gamma^n + t_{n-1} \Gamma^{n-1} + \cdots + t_1 \Gamma + t_0 = 0,$$

we have

$$(D(\vec{t})\Gamma)^n + \frac{t_{n-1}D(\vec{t})}{t_n}(D(\vec{t})\Gamma)^{n-1} + \cdots + \frac{t_1 D(\vec{t})^{n-1}}{t_n}(D(\vec{t})\Gamma) + \frac{t_0 D(\vec{t})}{t_n} = 0;$$

hence $t_n \mid D(\vec{t})$, and thus $I = t_n \mathbb{Z}[\vec{t}]$. ■

This proposition shows that if Γ is a root of the generic polynomial, then its denominator and leading coefficient are equal. Since the minimal polynomial over \mathbb{Z} of any algebraic number may be obtained by specializing the coefficients of the generic polynomial, it is tempting to conclude that this confirms the result of Arno et al.

Question. *Is it possible to find an uncountable set E and a property (\mathbf{P}) satisfying the conditions stated in the introduction?*

If the indexing set I is countable, then the set F of elements of E satisfying (\mathbf{P}) , being a countable union of finite sets, is itself countable. So its density in E is 0. The question remains for E and I uncountable. Our feeling is that the answer is negative.

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References

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